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On Exactly Solvable Lattice Models and Their Mathematical Properties

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Abstract

This paper provides a comprehensive review of our previously obtained results on exactly solvable lattice models, with a primary focus on the Abelian sandpile model, dimer model, loop-erased random walks and their connections to the enumeration of spanning trees.
Keywords: Exactly solvable lattice models, Non-equilibrium systems, Self-organized criticality.
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1. Introduction

The study of exactly solvable lattice models plays a fundamental role in the development of equilibrium and nonequilibrium statistical theories. These models are typically governed by relatively simple local dynamical rules, yet they often exhibit nontrivial critical behavior of observable quantities.

Investigations of such models allow one to understand, through tractable examples, mechanisms underlying critical behavior in realistic systems, both at and away from equilibrium.

A key equilibrium lattice model is the Ising model [1, 2], originally introduced to describe critical properties near phase transitions. The exact solution of the Ising model on the two-dimensional square lattice was first given by Onsager [3], demonstrating the possibility of second-order phase transitions in systems with short-range interactions. The model reveals critical exponents that diverge from those derived using perturbative or mean-field approximations.

Beyond the Ising model, further models such as vertex models and generalized spin systems have emerged. One prominent generalization is the Q-state Potts model [4], where the order of the phase transition depends on the parameter Q. Exact solutions exist only for specific parameter values in two dimensions.

Various exact methods and the analysis of two-dimensional models are discussed comprehensively in multiple monographs [5, 6, 7].

Alternative formulations of the Ising model have also appeared. The Pfaffian formulation by Hurst and Green [8], later recast by Kasteleyn in terms of dimer coverings [9, 10], connected the model to the dimer problem introduced by Fowler and Rushbrooke [11]. Significant exact results followed in the 1960s [9, 12, 13, 14], with the field expanding further through later works [15, 16, 17, 18].

A mapping between dense dimer packings and spanning trees was established by Temperley [19] and extended to general planar graphs [20]. For a detailed review of the dimer model and its relation to loop-erased random walks, spanning trees and Green functions of discrete Laplacian, see the work by Kenyon [21]. This correspondence persists even with the inclusion of monomers, leading to models of spanning webs [22, 23, 24].

Fortuin and Kasteleyn [25, 26] demonstrated that lattice models constitute a broad class of graph-theoretic problems, intersecting probability, combinatorics, and other domains. They introduced the random cluster model, thee partition function of which reduces, under parameter variations, to the Tutte polynomial [27, 28], spanning tree enumeration [29], percolation [30], the Potts model, and the Ashkin-Teller model.

Kirchhoff's matrix-tree theorem [31] provides a means to compute network resistances and connects these ideas to the enumeration of spanning trees.

Interest has grown in nonequilibrium exactly solvable lattice models. Criticality in such systems typically arises without fine-tuning of parameters. Correlation functions decay exponentially away from criticality:

$$\mathcal{R}(r) \sim \exp -r/\xi, \tag{1}$$

where ξ denotes the correlation length. At criticality, a power-law decay appears:

$$\mathcal{R}(r) \sim \frac{1}{r^{\alpha}}. \tag{2}$$

Critical behavior is marked by scale invariance and the absence of characteristic length scales. Below the critical temperature, equilibrium systems may exhibit spontaneous ordering emerging purely from internal dynamics.

In nonequilibrium settings, systems often evolve into a subset of configurations the recurrent set from which escape is impossible, similar to attractors in deterministic systems. This leads to a form of internal ordering. The resulting framework, where critical behavior arises self-consistently, is termed *self-organized criticality* (SOC).

SOC systems, while governed by local rules, exhibit highly nonlocal behavior. A conjecture posits that this nonlocality may introduce logarithmic corrections in asymptotic correlation functions:

$$\mathcal{R}(r) \sim \frac{(\log r)^{\beta}}{r^{\alpha}}. \tag{3}$$

These systems are typically dissipative and open: they maintain stationary macroscopic flows and convert all externally supplied energy into overcoming internal friction.

In 1987, Bak, Tang, and Wiesenfeld proposed a theory of SOC [32], asserting that many systems naturally evolve toward criticality. Perturbations can trigger avalanches of all scales. A canonical model in this context is the Abelian Sandpile Model (ASM) [32, 33, 34, 35], a nonlinear stochastic system defined on graphs. Each vertex holds a discrete height. A local update (addition of a grain) may trigger a cascade of topplings governed by deterministic rules, forming a cellular automaton. Only a subset of configurations recurrent configurations emerge asymptotically. The stationary measure is uniform over this set, and transient states have zero measure.

The Abelian property ensures that the order of relaxation steps does not affect the final state, making ASM analytically tractable. The burning algorithm [35, 36, 37, 38] provides a test for recurrence and constructs the associated spanning tree. Thus, a bijection exists between recurrent states and spanning trees.

Numerous models have been proposed to describe SOC: sand models [39], earthquake models [40, 41], forest-fire models [42, 43], and others.

Exact solvable ASM variants on regular lattices showed that renormalization group approaches may yield incorrect predictions [44]. Square lattices are especially relevant due to their rich nonlocal structure.

The source of nonlocality lies in recurrence testing, which involves global height configuration analysis.

Interestingly, probabilities of local observables (e.g., a specific height at a site) are determined by nonlocal features of uniform spanning trees and dimer-monomer coverings. For example, the number of neighboring predecessors in a spanning tree is a nonlocal quantity, computable via path enumeration in the tree.

In the ASM, the probability P_1 for height 1 and the asymptotics of the two-point function

$$\sigma_{11}(r) = P_{11}(r) - P_1^2, \quad r \gg 1$$

were computed by Majumdar and Dhar [36], using the spanning tree representation and Kirchhoff's theorem.

Higher height probabilities (P_2, P_3, P_4) were addressed by Priezzhev [38] via Θ-graph enumeration, involving integrals of singular functions derived from 4×4 determinants. Later, Ruelle [45] simplified them to a double integral.

This confirmed Grassberger's Monte Carlo hypothesis [39, 46] that the average height in the ASM is a rational number, 25/8.

2. Review of Our Results

In [23], the close-packed dimer model (domino tiling) on a two-dimensional square lattice with a single vacancy at the center was analytically studied. By generalizing the spanning tree representation to spanning webs, determinantal expressions were derived for random variables describing dimer mobility. In the thermodynamic limit of large lattices, these expressions reduce to the computation of Toeplitz determinants and their minors. The exact probabilities for the vacancy to be strictly jammed and other diffusion characteristics were calculated. Their numerical values agree with the numerical results and conjectures presented in [22].

Asymptotic expressions for the probability distributions of height variables and their two-point correlation functions in the Abelian sandpile model on the two-dimensional square lattice were analytically obtained in [47, 48, 49, 50]. These expressions exhibit logarithmic behavior previously predicted by logarithmic conformal field theory. It was shown that the distribution of height probabilities is directly related to the return probability of a loop-erased random walk (LERW) passing through a neighboring vertex of the starting point. This connection was rigorously established via a mapping to a local monomer-dimer model

[51, 52]. Although methods for computing these quantities and their optimizations had been developed, and high-precision simulations were conducted, exact values remained conjectural for many years [45, 53, 54, 55].

In [50], the fixed-energy sandpile model with conservative vertices (closed boundaries) on the square lattice was studied, and a relation between its threshold density and the stationary density of the Abelian sandpile model with open (dissipative) boundaries was explored. While the minimal height initial state had previously been considered [56, 57], the authors of [50] generalized this by considering negative initial heights. A conjecture was proposed stating that the difference between the threshold and stationary densities tends to zero as the absolute value of the negative initial height tends to infinity. This conjecture was proven in [58], where a formal theory and a general theorem were established.

3. Conclusion

In this work, we summarized our recent results on several exactly solvable lattice models and highlighted their shared mathematical structures. The connections among the Abelian sandpile model, dimer coverings, loop-erased random walks, and spanning tree enumeration reveal a unifying combinatorial framework that continues to motivate further analytical and computational developments.

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Ճշգրիտ լուծվող ցանցային մոդելները և դրանց մաթեմատիկական հատկությունները

Վահագն Ս. Պողոսյան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ, Երևան, Հայաստան
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Ամփոփում

Այս աշխատանքը ներկայացնում է ճշգրիտ լուծվող ցանցային մոդելների վերաբերյալ մեր կողմից նախկինում ստացված արդյունքների համապարփակ ակնարկ՝ հիմնական ուշադրությունը կենտրոնացնելով աբելյան ավազահատիկային մոդելի, դիմերային մոդելի, ջնջված ցիկլերով պատահական դեգերման և ծածկող ծառերի թվարկման խնդիրների հետ նրանց կապերին:

Բանալի բառեր՝ ճշգրիտ լուծվող ցանցային մոդելներ, անհավասարակշիռ համակարգեր, ինքնակազմակերպված կրիտիկականություն:

Точно решаемые решёточные модели и их математические свойства

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Аннотация

В данной работе представлен комплексный обзор ранее полученных нами результатов по точно решаемым решёточным моделям, с основным вниманием к абелевой модели песка, димерной модели, случайным блужданиям со стертыми циклами и их связям с задачами перечисления покрывающих деревьев.

Ключевые слова: точно решаемые решёточные модели, неравновесные системы, самоорганизованная критичность.