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Regularly Varying of the Normalizing Constants in the Theorem of Convergence to a Positive Stable Distribution

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Abstract

This article examines the behavior of the normalizing constants in V. Feller's theorem on the convergence of distributions for sums of independent, identically distributed random variables with heavy tails at infinity. It is proved that, in this setting, the normalizing constant is regularly varying at infinity.
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1. Introduction

We consider a sequence of independent, identically distributed random variables with the distribution function $F(x)$. Suppose that for $x \to +\infty$, an asymptotic relation is executed:

$$1 - F(x) \sim \frac{x^{-\alpha} L(x)}{\Gamma(1-\alpha)}, \tag{1}$$

where $0 < \alpha < 1$, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, $L(x)$ - slowly varying function at infinity (SVFI), i.e., a positive function defined for $(0, \infty)$ and for each $x > 0$ fulfills the condition

$$\lim_{t \to +\infty} \frac{L(tx)}{L(t)} = 1.$$

Subsequently, according to Theorem 2 (see [1], XIII.6, p. 448), if F is the probability distribution, concentrated on $(0, \infty)$ and such that upon $n \to \infty$

$$F^{n*}(a_n x) \to G(x), \tag{2}$$

(at points of continuity), where $F^{n*}(\cdot)$ - n-fold convolution of distribution F with itself, while G is the proper distribution, not concentrated at one point and if the type of distribution F is (1), a_n variates in standard measure may be selected in a way that

$$\frac{nL(a_n)}{a_n^\alpha} \to 1. \tag{3}$$

In this case, the asymptotic relation (2) is executed along with the distribution of probabilities $G = G_\gamma$, where G_α is a stable distribution with $0 < \alpha < 1$, parameter focused on $(0, \infty)$ having Laplace-Stieltjes transform $e^{-\lambda^\alpha}$.

2. The Behavior of the Normalizing Constants in V. Feller's Theorem at Infinity

The positive function R is called (accurately) regularly varying at infinity if it is measurable on the $[A, \infty)$, $A > 0$ semiaxis and there exists such a number as $\alpha \in (-\infty, +\infty)$, which for a certain $x > 0$

$$\lim_{t\to+\infty} ((R(xt)/(R(t)) = x^\alpha.$$

Meanwhile, α is called the order (indicator) of the function R.

Suppose that $a_n = n^{1/\alpha}\varphi(n)$ and find out what features shall possess function $\varphi(n)$ in order to execute asymptotic (3).

By plugging in (3) an equation for a_n, we will deduce an equivalent (3) relation:

$$L\left(n^{1/alpha}\varphi(n)\right) \sim \varphi^\alpha(n),$$

or in a more general form:

$$L\left(t^{1/alpha}\varphi(t)\right) \sim \varphi^\alpha(t). \tag{4}$$

Consider the following relation:

$$R_t(x) = \frac{L((xt)^{1/\alpha}\varphi(tx))}{L(t^{1/alpha}\varphi(t))}. \tag{5}$$

By virtue of asymptotic relation (4) upon $t \to +\infty$ out of (5), it follows that

$$R_t(x) \sim \left(\frac{\varphi(tx)}{\varphi(t)}\right)^\alpha. \tag{6}$$

In ([2], p. 10), the following is proved:

Theorem 1. *(On the introduction of SVFI). If funcion L, defined on semiaxis $[A, +\infty)$, $A > 0$ – SVFI, such number $B \geq A$ will be found so that for all $x \geq B$ occurs the following representation:*

$$L(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(u)}{u}\, du\right\}, \tag{7}$$

where η- limited measurable function on $[B, +\infty)$ is such that

a) $\eta(x) \to c$ ($|c| < \infty$) and

b) $\varepsilon(x)$ - continuous function on $[B, +\infty)$ is such that $\varepsilon(x) \to 0$ in case of $x \to +\infty$.

Since L is SVFI, therefore using the relation (5), it is not complicated to deduce the following equation for $R_t(x)$:

$$R_t(x) = \exp\{\eta((tx)^{1/\alpha}\varphi(tx)) - \eta((t)^{1/\alpha}\varphi(t))\} \cdot \exp\left\{\int_{t^{1/\alpha}\varphi(t)}^{(tx)^{1/\alpha}\varphi(tx)} \frac{\varepsilon(y)}{y}\,dy\right\}. \quad (8)$$

By introducing the notation $a_t(x) = \frac{\varphi(tx)}{\varphi(t)}$, the expression (7) will be transformed into the following type:

$$R_t(x) = \exp\{\eta((tx)^{1/\alpha}\varphi(tx)) - \eta((t)^{1/\alpha}\varphi(t))\} \cdot \exp\left\{\int_{1}^{x^{1/\alpha}a_t(x)} \frac{\varepsilon\left(\varepsilon\left(t^{1/\alpha}\varphi(t)z\right)\right)}{z}\,dz\right\}. \quad (9)$$

In the case of $t \to +\infty$, the first factor in the right-hand part of the relation (8) by virtue of condition b) of Theorem 1, tends to unity. Therefore, upon the availability of sufficiently high t

$$R_t(x) \sim \exp\left\{\int_{1}^{x^{1/\alpha}a_t(x)} \frac{\varepsilon\left(s\left(t^{1/\alpha}\varphi(t)y\right)\right)}{y}\,dy\right\}. \quad (10)$$

Theorem 2. *In case of any $x > 0$, the following equation is true:*

$$\lim_{t\to+\infty} x^{1/\alpha}a_t(x) = 1.$$

Proof. It shall firstly be proved that $\overline{\lim}_{t\to+\infty}a_t(x) \nrightarrow +\infty$ for all $x \in (0, +\infty)$. Suppose that the contrary takes place: then for each $x > 0$, there exists a sufficiently high $t_0 = t_0(x)$, that in the case of all $t > t_0$, the following condition is executed:

$$x^{1/\alpha}a_t(x) > 1. \quad (11)$$

Further, condition b) means that for any $\delta > 0$, there exists $y_0 = y_0(\delta)$, such that for all $y > y_0$ occurs the the following inequality:

$$\varepsilon(y) < \delta. \quad (12)$$

Besides, since $t^{1/\alpha}\varphi(t) \to +\infty$ in case of $t \to +\infty$, we will select $t_1 \geq t_0$ such that upon $t > t_1$ inequality $t^{1/\alpha}\varphi(t) > y_0$ is executed by virtue of selecting t_0 and condition $z \geq 1$ apparent from (12), uniformly in z follows the inequality $\varepsilon(zt^{1/\alpha}\varphi(t)) < \delta$. Therefore, after uncomplicated transformation, the following inequality is deduced:

$$\exp\left\{\int_{1}^{x^{1/\alpha}a_t(x)} \frac{s\left(t^{1/\alpha}\varphi(t)y\right)}{y}\,dy\right\} \leq x^{\delta/\alpha}a_t^{\delta}(x). \quad (13)$$

On the other hand, by virtue of asymptotic relation (4) in the case of $t \to +\infty$, the following is concluded:

$$R_t(x) = \frac{L\left((xt)^{1/\alpha}\varphi(tx)\right)}{L\left(t^{1/\alpha}\varphi(t)\right)} \sim \left(\frac{\varphi(tx)}{\varphi(t)}\right)^{\alpha} = a_t^{\alpha}(x). \quad (14)$$

That's the inequality (11) from which we deduce the following:

$$x^{\delta/\alpha}a_t^{\delta}(x) \geq a_t^{\alpha}(x).$$

By selecting $\delta < \alpha$ from the previous inequality, we have the following:

$$x^{\delta/\alpha} \geq a_t^{\alpha-\delta(x)}. \tag{15}$$

Upon fixing $x > 0$, the left-hand side of (13) is limited, while the right-hand side by the virtue of limitation $\alpha - \delta > 0$ for $t \to +\infty$ tends to infinity, resulting in a contradiction. Thus, it can be concluded from (10) that for any $x > 0$, the following inequality holds:

$$\lim_{t\to+\infty} x^{1/\alpha} a_t(x) \leq 1. \tag{16}$$

Let's demonstrate that $a_t(x) \nrightarrow 0$ in the case of $t \to +\infty$. We'll also conduct the proof by an indirect proof method. Assume that for each $x > 0$ there exists such $t' = t'(x)$, that for all $t > t'$, the following condition is satisfied:

$$x^{1/\alpha} a_t(x) < 1. \tag{17}$$

Simultaneously $t'' > \max(t', t_1)$ may be taken as high that

$$x^{1/\alpha} a_t(x) \cdot t^{1/\alpha}\varphi(t) = (xt)^{1/\alpha}\varphi(xt) > y_0,$$

where y_0 is defined in (11).

Taking into consideration the above, it is not difficult to prove that

$$\begin{aligned} \exp\left\{\int_1^{x^{1/\alpha}a_{t(x)}} \frac{\varepsilon\left(t^{1/\alpha}\varphi(t)y\right)}{y}\,dy\right\} &= \exp\left\{-\int_{x^{1/\alpha}a_{t(x)}}^1 \frac{\varepsilon\left(t^{1/\alpha}\varphi(t)y\right)}{y}\,dy\right\} \\ &\geq \exp\left\{-\delta \ln z|_{x^{1/\alpha}a_{t(x)}}^1\right\} \\ &= a_t^\delta(x) \cdot x^{\delta/\alpha}. \end{aligned} \tag{18}$$

On the other hand, for all $x > 0$ upon sufficiently high t from (14), we have the following:

$$R_t(x) \sim a_t^\alpha(x) \geq a_t^\delta(x) \cdot x^{\delta/\alpha}.$$

By selecting $\delta < \alpha$, in (12) we will have the following:

$$a_t(x) \geq x^{\frac{\delta}{\alpha(\alpha-\delta)}} > 0,$$

that in the case of $t \to +\infty$ contradicts our assumption, i.e., the condition (17) is inexecutable. Thus, Theorem 2 is proved. ■

Thereof, it follows that for all $x > 0$ $\lim_{t\to+\infty} R_t(x) = 1$, while from relation (6) it is concluded that function $\varphi(t)$ is SVFI.

Thus, the following is proven:

Theorem 3. *If conditions (1) – (3) are executed, the norming quantity a_n is a regularly varying function at infinity with the parameter $1/\alpha$.*

3. Conclusion

If F is the distribution of probabilities, concentrated on $(0, \infty)$, for which in case of $x \to +\infty$ asymptotic relation (1) is executed and G_α is a stable distribution with the parameter $0 < \alpha < 1$ concentrated on $(0, \infty)$, then

$$F^{n*}\left(n^{1/\alpha}\varphi(n) \cdot x\right) \to G_\alpha(x),$$

where $\varphi(\cdot)$ is SVFI connected with SVFI $L(\cdot)$ by the following asymptotic relation

$$L\left(n^{1/\alpha}\varphi(n)\right) \sim \varphi^\alpha(n).$$

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Զուգամիտության թեորեմում նորմալացնող հաստատունների կանոնավոր փոփոխությունը դեպի դրական կայուն բաշխում

Ռոբերտ Ն. Չիտչյան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ, Երևան, Հայաստան
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Ամփոփում

Այս հոդվածն ուսումնասիրում է նորմալացնող հաստատունների վարքագիծը Վ. Ֆելլերի թեորեմում՝ կապված անկախ, նույնականորեն բաշխված պատահական փոփոխականների գումարների համար բաշխումների զուգամիտության հետ, որոնք ունեն եծանրե պոչեր անվերջության մեջ: Ցույց է տրվում, որ այս համատեքստում նորմալացնող հաստատունը կանոնավոր կերպով փոփոխվում է անվերջության մեջ:

Բանալի բառեր՝ ապահովագրություն, պատահական փոփոխական, կանոնավոր կերպով փոփոխվող ֆունկցիա, դանդաղ փոփոխվող ֆունկցիա, կայուն բաշխում:

Регулярное изменение нормализующих констант в теореме сходимости к положительному устойчивому распределению

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Аннотация

В данной статье рассматривается поведение нормирующих констант в теореме В. Феллера о сходимости распределений сумм независимых одинаково распределенных случайных величин с "тяжелыми" хвостами на бесконечности. Демонстрируется, что в данном контексте нормирующая константа регулярно меняется на бесконечности.

Ключевые слова: страхование, случайная величина, регулярно меняющаяся функция, медленно меняющаяся функция, устойчивое распределение.