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On the Proof of the Existence of Nontotal Partial Degree and on the Turing Degree of Representative of This Partial Degree

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Abstract

The ordering of *e*-degrees (of total functions) is known to be isomorphic to the ordering of *T*-degrees. It is possible to form equivalence classes with respect to $=_e$ and in the set of all functions (not necessarily total). The resulting *e*-degrees are called *partial degrees*.

In H. Rogers' *Theory of Recursive Functions and Effective Computability* [1], a proof of the existence of a non-total partial degree is given along with a corollary to this theorem.

The article contains a modification of the proof of the theorem given above, which allows us to significantly strengthen the results of the corollary, namely to prove that $(\exists\psi)[\ \psi$ is not partial computable $\ \&\ \ \psi \leq_T \mathbf{0}'\ \&\ \ (\forall f)[f \leq_e \psi \Rightarrow f$ is computable]] (in the above-mentioned corollary, it is noted that the constructed function is only computably enumerable in $\mathbf{0}'$).

Keywords: *e*-reducibility, partial degree, partial computable function, Turing degree.

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1. Introduction

Formal definitions of many concepts mentioned in the Introduction will be given in the Preliminary section.

The concept of enumeration reducibility was introduced in the works of Friedberg and Rogers [2], Myhill [3] and Selman [4]. Informally, $A \subseteq \omega$ (where ω is the set of the nonnegative integers), is enumeration reducible to $B \subseteq \omega$ if there is a uniform way to compute an enumeration of A from an enumeration of B.

If a partial degree contains at least one total function, it is called total. The total degrees therefore constitute a subordering of the partial degrees.

The structure of enumeration degrees $\mathcal{D}_e$ is an upper semi-lattice with the least upper bound induced by effective join operation $A \oplus B$ and the least element $\mathbb{0}_e$, the degree of all computably enumerable sets. The relationship between the following three reducibilities: *e*-reducibility, *T*-reducibility and relative computable enumerability (c.e. in) is expressed using a proposition

$$A \leq_T B \Leftrightarrow A \oplus \bar{A} \text{ is } B\text{-c.e.} \Leftrightarrow A \oplus \bar{A} \leq_e B \oplus \bar{B}.$$

Myhill [3] used this relationship to define a natural embedding of Turing degrees into enumeration degrees. He proved that the embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \bar{A})$, preserves the order and the least upper bound.

Research in the field of e-degrees has continued over the past decades. Among the latest works, we can note [5], [6], and [7].

In [1], the existence of a nontotal partial degree is proved, along with a corollary of this theorem. In the Results section of this article, a modified proof of the aforementioned theorem is presented, which substantially strengthens the results of its corollary.

2. Preliminaries

Notations. In this section, we shall give the necessary definitions. We shall use the notions and terminology introduced in Rogers [1], and Soare [8].

We deal with sets and functions over the nonnegative integers $\omega = \{0,1,2,\cdots\}$.

Let φ_e be the e^{th} partial computable function in the standard listing (see [8], p.15, p.25).

If $A \subseteq \omega$ and $e \in \omega$, let $\Phi_e^A(x) = \Phi_e(A{:}\,x) = \{e\}^A(x)$ (see [8], pp. 48-50).

χ_A denotes the characteristic function of A, which is often identified with A and written simply as $A(x)$.

We write $\varphi_{e,s}(x) = y$ if $x, y, e < s$ and y is the output of $\varphi_e(x)$ in $< e$-steps of the Turing program P_e. If such a y exists, we say $\varphi_{e,s}(x)$ *converges*, which we write as $\varphi_{e,s}(x)\downarrow$, and *diverges* ($\varphi_{e,s}(x)\uparrow$), otherwise.

Similarly, we write $\varphi_e(x)\downarrow$ if $\varphi_{e,s}(x)\downarrow$ for some s, and we write $\varphi_e(x)\downarrow = y$ if $\varphi_e(x)\downarrow$ and $\varphi_e(x) = y$, and similarly for $\varphi_{e,s}(x)\downarrow = y$ (see [8], pp.16-17).

$W_e = dom\, \varphi_e = \{x\colon \varphi_e(x)\downarrow\}$.

max (A) denotes the maximum element of a finite set A, if A is not $\emptyset$, and 0, otherwise.

$f \upharpoonright x$ denotes the restriction of f to arguments $y < x$, and $A \upharpoonright x$ denotes $\chi_A \upharpoonright x$.

Definition 1. $\tau(x,y) = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$.

It is known that τ is a computable one-one mapping of $\omega \times \omega$ onto ω.

We shall use $\langle x, y\rangle$ as an abbreviation for $\tau(x,y)$.

Definition 2. Given x, y, $<x, y>$ is the *ordered pair* consisting of x and y in that order.

Let R be any 2-ary relation. We say that R *is a single-valued* relation if for every x there exists at most one z such that $<x, z> \in R$.

A set A is *single-valued* if $\{<x, y> \mid \langle x, y\rangle \in A\}$ is a single-valued relation.

Definition 3. Let $K^A = \{x \mid \Phi_x^A(x)\downarrow\} = \{x \mid x \in W_x^A \downarrow\}$. K^A is called the *jump of* A and is denoted by A' (read as "A prime") (see [8], p. 53).

Definition 4. $\mathbf{0} = \deg(\emptyset) = \{B \mid B \text{ is computable}\}$,

$\mathbf{0}' = \deg(\emptyset')$, where $\emptyset' =_{dfn} K^{\emptyset}$ (see [8], p. 54).

Definition 5. a) Let *A join B*, written $A \oplus B$ be $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

b) Let $\{A_y\}_{y\in\omega}$ be any countable sequence of sets. Define the *infinite join*

$\oplus \{A_y\}_{y\in\omega} =_{dfn} \{\langle x, y\rangle \mid x \in A_y \ \& \ y \in \omega\}$ (see [8], p. 54).

Definition 6. *(i)* A sequence of (total) functions $\{f_s(x)\}$ *converges* (*pointwise*) to $f(x)$, written $f = \lim_s, f_s$, if for all x, $f_s(x) = f(x)$ for a.e. s (all but finitely many s).

(ii) A *modulus* (*of convergence*) for $\{f_s\}_{s\in\omega}$ is a function $m(x)$ such that for all s, if $s \geq m(x)$, then $f_s(x) = f(x)$ (Hence, $f_m(x) = f(x)$.) The *least modulus* is the function $m(x) = (\mu s)(\forall t \geq s)[f_s(x) = f(x)]$.

(iii) The sequence $\{f_s(x)\}_{s\in\omega}$ is *computable* if there exists a computable function $\hat{f}(x, s)$ such that $f_s(x) = \hat{f}(x, s)$ for all x, s.

Let $\{f_s(x)\}_{s\in\omega}$ be a computable sequence. Note that the least modulus is computable in any modulus. If $f = \lim_s, f_s$ and m is any modulus, then

$$f \leq_T m$$

because $f_{m(x)}(x) = f(x)$. However, $m \leq_T f$ usually fails even for the least modulus. Remarkably, if f has c.e. degree, then $m \leq_T f$ holds for some modulus m of a particular computable sequence, as we prove in the following lemma.

Modulus Lemma. *If A is c.e. set and* $f \leq_T A$, *then there exists a computable sequence* $\{f_s\}_{s\in\omega}$ *such that* $\lim_s f_s = f$, *and a modulus m of* $\{f_s\}_{s\in\omega}$, *which is computable in A.* (see [8], p.56)

Let us present the Limit Lemma along with its proof, since it will be used in proving our theorem.

Limit Lemma. *For any function* f, $f \leq_T A'$ *iff there exists an A-computable sequence* $\{f_s\}_{s\in\omega}$ (*i.e., an A-computable function* $\hat{f}(x, s) = f_s(x)$) *such that* $f = lim_s f_s$.

Proof. (⇒). Let $f \leq_T A'$. Now A' is c.e. in A. Hence, the A-computable sequence $\{f_s\}_{s\in\omega}$ exists by the Modulus Lemma relativized to A.

(⇐). Let $f = lim_s f_s$. Define

$$A_x = \{s : (\exists t)\, [s \leq t \ \& \ f(x) \neq f_{t+1}(x)]\}.$$

Now A_x is finite, and $B = \oplus_x A_x = \{(s, x) : s \in A_x\}$ is Σ_1^A and hence A-c.e., so $B \leq_T A'$.

Thus, given x, we can B-computably (and therefore A'-computably) compute the least modulus $m(x) = (\mu s)\, [s \notin A_x]$. Hence, $f \leq_T m\, A \leq_T B \oplus A \leq_T A'$.

In particular, $f \leq_T \mathbf{0}'$ if and only if $f = lim_s f_s$ for some computable sequence $\{f_s\}_{s\in\omega}$. This will be the most useful characterization of degrees below $\mathbf{0}'$. Since not all degrees below $\mathbf{0}'$ are c.e., the following corollary selects those that are.

Corollary of the Limit Lemma. *A function* f *has c.e. degree iff* f *is the limit of a computable sequence* $\{f_s\}_{s\in\omega}$, *which has a modulus* $m \leq_T f$ (see [8], p.57).

Definition 7. Given a finite set $A = \{x_1, x_2, \cdots, x_k\}$, where $x_1 < x_2 < \cdots < x_k$, the number $y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_k}$ is the *canonical index* of A. Let D_y denote a finite set with canonical index y, and D_0 denote $\emptyset$.

Definition 8. $A \subseteq \omega$ is an *enumeration reducible* to a set $B \subseteq \omega$ ($A \leq_e B$) if there is c.e. set W_z such that $A = \{n \mid (\exists e)[\langle n, e\rangle \in W_z \ \& \ D_e \subseteq B]\}$, where D_e is the e-th finite set in canonical enumeration.

Thus, any z and any B determine a unique corresponding A such that $A \leq_e B$ via z, namely $\{x \mid (\exists u)[\langle x, u\rangle \in W_z \ \& D_u \subseteq B]\}$. Hence, each z determines a total mapping from 2^ω to 2^ω. We call such mappings *enumeration operations* and denote the operator corresponding to z as Φ_z (see [1], pp.146-147).
Every *T*-degree (of total functions) is a subcollection of some partial degree. If a partial degree contains a *T*-degree (of total functions), we call it a *total degree* (see [1], p. 280).

3. Results

In [1], Theorem13.XVIII is proved (announced by Medvedev [9]): $(\exists\psi)$ [ψ *is not partial computable* & $(\forall f)[f \leq_e \psi \Rightarrow f$ *is computable*]] and the Corollary is presented: $(\exists\psi)$ [ψ is not partial computable and ψ is computably enumerable in $\mathbf{0}'$ and $(\forall f)[f \leq_e \psi \Rightarrow f$ is computable]] (see [1], pp. 280-281).

Let us prove the following Theorem.

Theorem 1. $(\exists\psi)$ [ψ *is not partial computable* & $\psi \leq_T \mathbf{0}'$ & $(\forall f)[f \leq_e \psi \Rightarrow f$ *is computable*]].

Proof. Note that we identify functions with their graphs and define $f \leq_e g$ if $\tau(f) \leq_e \tau(g)$.

Recall that $\Phi_n(\tau(\psi))$ is the enumeration operator of index n. We use the following notation in the proof.

$\Phi_n(\psi)$ abbreviates $\Phi_n(\tau(\psi))$. (Thus $\Phi_n(\psi)$ is a set that may not be single-valued.) If $\Phi_n(\psi)$ is single-valued, we also abbreviate $\tau^{-1}(\Phi_n(\psi))$ as $\Phi_n(\psi)$. (Thus, for example, we can write $f \leq_e \psi \Leftrightarrow (\exists n)[f = \Phi_n(\psi)]$.)

At each stage s, we construct finite segments $\psi_{m,s}$ such that $\max(dom\ \psi_{m,s}) < s$ & $m < s$. The function ψ will be called a *finite segment* if the domain of ψ is finite. A finite segment will be called a *monotone extension* of ψ if
$\psi \subset \tilde{\psi}$ and $(\forall x)(\forall y)[[x \in dom\ \psi$ and $y \in dom(\tilde{\psi} - \psi)] \Rightarrow x < y]$.

The construction will be such that for any s $(\forall m < s)(\forall n < s)[m < n \Rightarrow \psi_{n,s}$ is a monotone extension of $\psi_{m,s}]$ and for any m $(\exists s_0)(\forall s \geq s_0)(\psi_{m,s} = \psi_{m,s_0} =_{dfn} \psi_m)$ (since the construction uses the finite injury priority method).

We prove the theorem by obtaining the desired ψ as the union of a sequence of finite segments $\psi_0, \psi_1, \cdots$, where $m < n \Rightarrow \psi_n$ is a monotone extension of ψ_m.
Note that the construction is such that the set $\{\langle m, s, x, \psi_{m,s}(x)\rangle \mid \psi_{m,s}(x)\downarrow\}$ is computable.

As a result, we will have $(\forall x)(\exists m_0)(\exists s_0)(\forall m > m_0)(\forall s > s_0)\ \psi \restriction x = \psi_{m,s} \restriction x$.

In the process of constructing ψ, stages are implemented to ensure that for any n, the function $\Phi_n(\psi)$ will ultimately be either computable or nontotal (if $\Phi_n(\psi)$ is single-valued).

Let us denote the number n by $Q(k)$, if $k = 2n + 1 \vee k = 2n + 2$.
For any n, the finite segment ψ_n is intended to work with the operator $\Phi_{Q(n)}$ or the function $\varphi_{Q(n)}$.

Let $v_1(n) = \max\{k|\ k \leq n\ \&$ the actions taken (at stage n) concerning the function φ_k are not canceled$\}$ (i.e., remains *valid* at stage n).

Let $v_2(n) = \max\{k|\ k \leq n\ \&$ the actions taken (at stage n) concerning the operator Φ_k are not canceled$\}$ (i.e., remains *valid* at stage n).

If the actions taken (at stage n) concerning the operator Φ_k (the function φ_k) are canceled, we will briefly say, Φ_k (φ_k) is canceled (at stage n).

Stage 0. Let $\psi_0 = \emptyset$.

Stage $2n + 1$. *See* whether there exists $k \leq Q(v_1(2n))$ such that

$$\varphi_{k,2n}(\max(dom\, \psi_{2k,2n}) + 1) \uparrow\ \&\ \varphi_{2k,2n+1}(\max(dom\, \psi_{2k,2n}) + 1) \downarrow .$$

If so, then let k_0 be the least of such numbers and $p_0 = \max\big(dom\, \psi_{2k_0,2n}\big)$.
We set $\psi_{2k_0+1,2n+1} = \psi_{2k_0,2n} \cup \{<p_0, \varphi_{k_0,2n+1}(p_0)+1>\}$.
Then we cancel φ_k for all $k > k_0$ and Φ_k for all $k \geq k_0$.

If not, then let $\tilde{q} = \max(\{q\ |\varphi_q$ is not canceled at stage $2n\}$ and $p_1 = \max(dom\, \psi_{2\tilde{q}+1,2n}) + 1$.
We set $\psi_{2\tilde{q}+1,2n+1} = \psi_{2\tilde{q}+1,2n} \cup \{<p_1, 0>\}$.
(Stage $2n + 1$ ensures that $\psi \neq \varphi_n$; therefore, ψ cannot be partial computable.)

Stage $2n + 2$. Substage (a). *See* whether there exists $k \leq Q(v_2(2n + 1))$ such that there exists a monotone extension $\tilde{\psi}$ of the segment $\psi_{k,n+1}$ such that $\Phi_k(\tilde{\psi})$ is not single-valued and $\max\big(dom\, \tilde{\psi}\big) < 2n + 2$.
If so, then let k_0 be the least of such members. Then we set $\psi_{k_0,2n+2} = \tilde{\psi}$ and go to stage $2n + 3$. (In this case, the actions taken (at stage $2n + 2$, substage (a)) concerning the operators and functions with numbers greater than k_0 are canceled.) *If not*, we go to substage (b).

Substage (b). **Notation**. $\mathrm{Div}(m, s)$ means that there exist a number y and monotone extensions $\tilde{\psi}^1$ and $\tilde{\psi}^2$ of the segment $\psi_{m,s}$ such that the values of the functions $\Phi_{2m}\big(\tilde{\psi}^1\big)$ and $\Phi_{2m}\big(\tilde{\psi}^2\big)$ as partial functions, are defined and unequal for the argument y & $\max\big(dom\, \tilde{\psi}^1\big) < s + 1\ \&\ \max\big(dom\, \tilde{\psi}^2\ \big) < s + 1$.

See whether there exists a number $k \leq Q(v_2(2n + 1))$ such that $\mathrm{Div}(k, 2n + 1)$.

If so, then let k_1 be the least of such numbers. Then set $\psi_{k_1,2n+2} = \psi_{k_1,2n+1} \cup \{<z, 0>\}$, where z is the least of the numbers greater than all the elements of the domains of both the segment $\tilde{\psi}^1$ and the segment $\tilde{\psi}^2$. (In this case, the function $\Phi_{2k_1}(\psi)$ must be undefined at y, for, otherwise, ψ could be used together with $\tilde{\psi}^1$ or else $\tilde{\psi}^2$ to provide a monotone extension $\tilde{\psi}$ of the segment $\psi_{k_1,2n+1}$, for which $\Phi_{2k_1}(\tilde{\psi})$ is not single-valued, contrary to the result of substage (a)). In this case, the actions taken (at stage $2n + 2$, substage (b)) concerning the operators and functions with numbers greater than k_1, are canceled.
If not, then let $\psi_{k_1,2n+2} = \psi_{k_1,2n+1}$. (In this case, $\Phi_{2k_1}(\psi)$ must be computable, if total, for it can be effectively computed by enumerating all monotone extensions of $\psi_{k_1,2n+1}$ and putting them through Φ_{2k_1}.)

Note that according to the construction, at any stage n, the transition from the finite segment constructed at stage n to the finite segment that will be constructed at stage n+1 occurs effectively (uniformly over n).

As already noted, we have defined $\psi =_{dfn} \bigcup_{i\in\omega} \psi_i$ (the definition of ψ_i is given above).

Set $S_x = \{s|(\exists m)(s \leq m\ \&\ \psi_m \upharpoonright x\ \neq \psi_{m+1} \upharpoonright x)\}$.

Then S_x is finite, just like A_x in the proof of the Limit Lemma.

$\tilde{B} =_{dfn} \oplus\ S_x = \{\langle s, x\rangle|\ s \in S_x\}$.

$\tilde{B}$ is a computably enumerable set (similarly, the set B in the proof of Limit Lemma is A- computably enumerable).

Then, given x, we can $\tilde{B}$ -computably (and therefore $\mathbf{0}'$- computably) compute the function $\tilde{m}(x) =_{dfn} \{\mu x | s \notin S_x\}$. Hence, $\psi \leq_T \tilde{m} \leq_T \tilde{B} \leq_T \mathbf{0}'$. ■

4. Conclusion

In the above theorem from [1], when constructing the function ψ, to achieve nontotality of $\deg_e(\psi)$, actions were performed on the e-operators Φ_e (for any e), only one stage was sufficient to complete the necessary actions on each specific e-operator. In the proof given in this article, a much larger (but finite) number of stages may be required to complete the necessary actions on each specific e-operator.

This modification of the proof allows us to substantially strengthen the results of the corollary of the mentioned theorem. The results are presented in more detail in Section 3.

References

[1] H. Rogers Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[2] R. M. Friedberg and H. Rogers Jr., *Reducibility and completeness for sets of integers*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 5, pp. 117–125, 1959.

[3] J. Myhill, “Note on degrees of partial functions”, *Proceedings of the American Mathematical Society* 12, pp. 519–521, 1961.

[4] A. L. Selman, “Arithmetical reducibilities”*, I, Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 17, pp. 335–350, 1971.

[5] M. I. Soskova, *The theory of the enumeration degrees, definability, and automorphisms*, Contemporary Logic and Computing. Adrian Rezus, editor. College publications, pp.706-730, 2020.

[6] S. Lempp, Th. Slaman and M. I. Soskova, “Fragments of the theory of the enumeration degrees”, *Advances of Mathematics*, vol. 383, no. 4, pp.107686, 2021.

[7] H. Ganchev, I. Kalimullin, J. S. Miller, and M. I. Soskova, “A structural dichotomy in the enumeration degrees”, *J. Symb. Log.*, vol. 87, no. 2, pp. 527-544, 2022.

[8] R.I. Soare, *Recursively Enumerable Sets and Degree: A study of computable functions and computably generated sets*, Perspectives in Mathematical Logic, Springer-Verlag, 1987.

[9] Yu. T. Medvedev, *Degrees of difficulty of the mass problem* (in Russian), Doklady Academii Nauk SSSR, vol. 104, pp. 501-504, 1955.

Ոչ տոտալ մասնակի աստիճանի գոյության ապացույցի և այդ մասնակի աստիճանի ներկայացուցչի թյուրինգյան աստիճանի մասին

Արսեն Հ. Մոկացյան

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Ամփոփում

Հայտնի է, որ ամենուրեք որոշված ֆունկցիաների e-աստիճանների կարգավորումը իզոմորֆ է T-աստիճանների կարգավորմանը: Հնարավոր է ըստ $=_e$-ի համարժեքության դասեր ձևավորել նաև բոլոր ֆունկցիաների (պարտադիր չէ ամենուրեք որոշված) ամբողջության մեջ: Ստացված e-աստիճանները անվանվում են *մասնակի աստիճաններ*:

Հ. Ռոջերսի «Ռեկուրսիվ ֆունկցիաների տեսություն և էֆեկտիվ հաշվարկելիություն» գրքում [1] ներկայացված են ոչ տոտալ մասնակի աստիճանի գոյության ապացույցը և այդ պնդման հետևությունը:

Տվյալ հոդվածում ներկայացված է վերոհիշյալ թեորեմի ապացույցի մոդիֆիկացիան, ինչը թույլ է տալիս էականորեն ուժեղացնել թեորեմի հետևության արդյունքները, այսինքն՝ ապացուցել, որ $(\exists\psi)$[ψ-ն մասնակի հաշվարկելի չէ & $\psi \leq_T \mathbf{0}'$ & $(\forall f)[f \leq_e \psi \Rightarrow f$ հաշվարկելի է]] (վերոնշյալ հետևության մեջ նշվում է, որ կառուցված ֆունկցիան ընդամենը հաշվարկելիորեն թվարկելի է ըստ $\mathbf{0}'$-ի):

Բանալի բառեր՝ e-հանգեցում, մասնակի աստիճան, մասնակի հաշվարկելի ֆունկցիա, թյուրինգյան աստիճան:

О доказательстве существования нетотальной частичной степени и о тьюринговой степени представителя этой частичной степени

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Аннотация

Известно, что упорядочение e-степеней всюду определенных функций изоморфно упорядочению T-степеней. Можно образовать классы эквивалентности относительно $=_e$ и в совокупности всех функций (не обязательно всюду определенных). Полученные при этом e-степени называются *частичными степенями*.

В книге Х. Роджерса, “*Теория рекурсивных функций и эффективная вычислимость”,* [1] дано доказательство существования нетотальной частичной степени и приведено следствие из этой теоремы.

Статья содержит модификацию доказательства теоремы, приведенной выше, которая позволяет существенно усилить результаты следствия, а именно доказать, что $(\exists\psi)$[функция ψ не является частично рекурсивной & $\psi \leq_T \mathbf{0}'$ & $(\forall f)[f \leq_e \psi \Rightarrow$ функция f рекурсивна]] (в приведенном выше следствии отмечено, что построенная функция всего лишь рекурсивно перечислима относительно **0**').

Ключевые слова: e-сводимость, частичная степень, частично рекурсивная функция, тьюрингова степень.