Mathematical Problems of Computer Science 49, 35–40, 2018.

Degree Sequences and Dominating Cycles in 2-Connected Graphs

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Abstract

Let G be a graph on n vertices and minimum degree δ with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. The minimum degree sum of two nonadjacent vertices in G is denoted by σ_2. Let c be the circumference - the order (the number of vertices) of a longest cycle, and p be the order of a longest path in G. In 1952, Dirac proved: (1) if G is a 2-connected graph, then $c \geq \min\{n, 2d_1\}$; (2) every graph with $d_1 \geq \frac{n}{2}$ is hamiltonian. Recently, these results were improved by Nikoghosyan in terms of degree sequences: (3) if G is a 2-connected graph, then $c \geq \min\{n, d_\delta + d_{\delta+1}\}$; (4) every graph with $d_\delta + d_{\delta+1} \geq n$ is hamiltonian. In this paper we present the dominating cycle versions of these theorems: (i) if G is a 2-connected graph, then either $c \geq d_\delta + d_{\sigma_2}$ or $c \geq p - 1$ (that is G has a dominating cycle); (ii) every 2-connected graph with $d_\delta + d_{\delta+1} \geq p - 1$ has a dominating cycle. The results are sharp.

Keywords: Hamilton cycle, dominating cycle, circumference, minimum degree, degree sums, degree sequence.

1. Introduction

We consider only finite undirected graphs with neither loops nor multiple edges. A good reference for any undefined terms is [1].

The set of vertices of a graph G is denoted by $V(G)$, and the set of edges by $E(G)$. Let n be the order (the number of vertices) of G, c the order of a longest cycle (called circumference) in G and p the order of a longest path. The minimum degree sum of two nonadjacent vertices in G is denoted by σ_2. In particular, the minimum degree in G is denoted by δ. Let $d_1, d_2, ..., d_n$ be the degree sequence in G with $\delta = d_1 \leq d_2 \leq ... \leq d_n$. We use $N(v)$ to denote the set of all neighbors of a vertex v and $d(v) = |N(v)|$ to denote the degree of vertex v. A graph G is hamiltonian if G contains a Hamilton cycle, that is a simple spanning cycle. A cycle C of G is called a dominating cycle if every edge of G has at least one of its end vertices on C, or, equivalently, if $G - V(C)$ contains no edges.

We write a cycle Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. For $x \in V(Q)$, we denote the successor and the predecessor of x on $\overrightarrow{Q}$ (if such vertices exist) by x^+ and x^-, respectively. For $U \subseteq V(Q)$, we denote $U^+ = \{u^+ | u \in U\}$ and $U^- = \{u^- | u \in U\}$. We say that the vertex

z_1 precedes the vertex z_2 on a path $\overrightarrow{Q}$ if z_1, z_2 occur on $\overrightarrow{Q}$ in this order and indicate this relationship by $z_1 \prec z_2$. We will write $z_1 \preceq z_2$ when either $z_1 = z_2$ or $z_1 \prec z_2$.

Let $\overrightarrow{P} = v_1v_2...v_p$ be a longest path in G. Clearly, $N(v_1) \cup N(v_p) \subseteq V(P)$. A vine of length m on P is a set

$$\{L_i = w_i \overrightarrow{L}_i z_i : 1 \leq i \leq m\}$$

of internally-disjoint paths such that

(a) $V(L_i) \cap V(P) = \{w_i, z_i\}$ $(i = 1, ..., m)$,
(b) $v_1 = w_1 \prec w_2 \prec z_1 \preceq w_3 \prec z_2 \preceq w_4 \prec ... \preceq w_m \prec z_{m-1} \prec z_m = v_p$ on P.

The following result guarantees the existence of at least one vine on $\overrightarrow{P}$ in a 2-connected graph.

The Vine Lemma [2]. If G is a 2-connected graph and P a path in G, then there is at least one vine on P.

In 1952, Dirac [2] obtained the first lower bound for the circumference for 2-connected graphs and the first sufficient condition for Hamilton cycles in terms of minimum degree δ.

Theorem A [2]. *If G is a 2-connected graph, then $c \geq \min\{n, 2\delta\} = \min\{n, 2d_1\}$.*

Theorem B [2]. Every graph with $\delta = d_1 \geq \frac{n}{2}$ is hamiltonian.

Theorems A and B were improved in [3] in terms of degree sequences.

Theorem C [3]. *If G is a 2-connected graph, then $c \geq \min\{n, d_\delta + d_{\delta+1}\}$.*

Theorem D [3]. *Every graph with $d_\delta + d_{\delta+1} \geq n$ is hamiltonian.*

In this paper we present the dominating versions of Theorems C and D.

Proposition 1 [4]. *Let G be a connected graph with $c \geq p - 1$. Then every longest cycle in G is a dominating cycle.*

Theorem 1. *If G is a 2-connected graph, then $c \geq \min\{p - 1, d_\delta + d_{\sigma_2}\}$.*

The next result follows from Theorem 1 immediately as a sufficient condition for the existence of a dominating cycle.

Theorem 2. *If G is a 2-connected graph with $d_\delta + d_{\sigma_2} \geq p - 1$, then $c \geq p - 1$.*

If $G = \overline{K}_{\delta+1} + K_\delta$, then $d_\delta = \delta$, $d_{2\delta} = d_{\sigma_2} = 2\delta = \sigma_2$ and $c = 2\delta = \sigma_2 = p - 1$. This graph example shows that the conclusion "either $c \geq d_\delta + d_{\sigma_2}$ or $c \geq p - 1$" in Theorem 3 cannot be replaced by "either $c \geq d_\delta + d_{\sigma_2}$ or $c \geq p$". Next, let $G = \delta K_2 + K_{\delta-1}$. Then $d_\delta = d_{2\delta} = d_{\sigma_2} = \delta$, $d_{2\delta+1} = d_{\sigma_2+1} = 3\delta - 2$ and $c = 3\delta - 3 = p - 2$. This graph example shows that the conclusion "either $c \geq d_\delta + d_{\sigma_2}$ or $c \geq p - 1$" in Theorem 1 cannot be replaced by "either $c \geq d_\delta + d_{\sigma_2+1}$ or $c \geq p-1$". Thus, Theorem 1 is best possible.

2. Proofs

Proof of Theorem 1. Let $\overrightarrow{P} = v_1 v_2 ... v_p$ be a longest path in G. Clearly,

$$N(v_1) \cup N(v_p) \subseteq V(P).$$

Assume that

$(a1)$ P is chosen so that $d(v_1)$ is maximum.

$(a2)$ P is chosen so that $d(v_p)$ is maximum subject to $(a1)$.

Let $x_1, x_2, ..., x_t$ be the elements of $N(v_1)$ occurring on $\overrightarrow{P}$ in a consecutive order, where $t = d(v_1) \geq \delta$. Next, let $y_1, y_2, ..., y_f$ be the elements of $N(v_p)$ occurring on $\overleftarrow{P}$ in a consecutive order. If either $x_t = v_p$ or $y_f = v_1$, then we can form a path longer than P, a contradiction. Hence, $x_t \neq v_p$ and $y_f \neq v_1$. Observe that for each $i \in \{1, 2, ..., t\}$,

$$x_i^- \overleftarrow{P} v_1 x_i \overrightarrow{P} v_p$$

is a longest path in G, implying that

$$N(x_i^-) \subseteq V(P) \quad (i = 1, 2, ..., t).$$

By a symmetric argument,

$$N(y_i^+) \subseteq V(P) \quad (i = 1, 2, ..., f).$$

Case 1. $x_t \preceq y_f$.
Let

$$\{L_i = w_i \overrightarrow{L}_i z_i : 1 \leq i \leq m\}$$

be a vine of minimal length m on $\overrightarrow{P}$. Since P is a longest path in G, we have $L_1, L_M \in E(G)$. Next, since m is minimal, we have $x_t \prec z_2$, $x_t \prec w_3$ and $w_{m-1} \prec y_f$, $z_{m-2} \prec y_f$. Choose $z_1^* \in V(P)$ such that $w_2 \prec z_1^*$ and $|V(w_2 \overrightarrow{P} z_1^*)|$ is minimal. Analogously, choose $w_m^* \in V(P)$ such that $w_m^* \prec z_{m-1}$ and $|V(w_m^* \overrightarrow{P} z_{m-1})|$ is minimal. Put

$$H = P \cup \bigcup_{i=2}^{m-1} L_i \cup \{v_1 z_1^*, v_p w_m^*\}.$$

By deleting the following paths

$$w_i \overrightarrow{P} z_{i-1} \quad (i = 3, 4, ..., m-1), \quad w_2 \overrightarrow{P} z_1^*, \quad w_m^* \overrightarrow{P} z_{m-1}$$

from H (except for their endvertices), we obtain a cycle C with at least $d(v_1) + d(v_p) + 1$ vertices. Since the vertices $x_1^-, x_2^-, ..., x_t^-, y_1^+, y_2^+, ..., y_f^+$ are pairwise distinct, we have

$$d(v_1) = \max\{d(x_1^-), d(x_2^-), ..., d(x_t^-), d(y_1^+), d(y_2^+), ..., d(y_f^+)\}$$

$$\geq \max\{d_1, d_2, ..., d_{t+f}\} = d_{t+f} = d_{d(v_1)+d(v_p)} \geq d_{\sigma_2},$$

$$d(v_p) = \max\{d(y_1^+), d(y_2^+), ..., d(y_f^+)\} \geq \max\{d_1, d_2, ..., d_f\} = d_f = d_{d(v_p)} \geq d_\delta,$$

implying that

$$c \geq d(v_1) + d(v_p) + 1 > d_\delta + d_{\sigma_2}.$$

Case 2. $y_f \prec x_t$.

Case 2.1. $N(v_1) \cap N^+(v_p) \neq \emptyset$.

Let $v \in N(v_1) \cap N^+(v_p)$, that is $v_1v, v_pv^- \in E(G)$. Since

$$v_1v\overrightarrow{P}v_pv^-\overleftarrow{P}v_1$$

is a cycle of order p, and G is connected, either $p < |V(G)|$, and we can form a path longer than P (a contradiction) or $p = |V(G)|$, implying that $c = p$.

Case 2.2. $N(v_1) \cap N^+(v_p) = \emptyset$.

Case 2.2.1. $N^-(v_1) \cap N^+(v_p) \neq \emptyset$.

Let $v \in N^-(v_1) \cap N^+(v_p)$, that is $z = x_i^- = y_j^+$ for some $i \in \{1, ..., t\}$ and $j \in \{1, ..., f\}$. Clearly,

$$v_1z^+\overrightarrow{P}v_pz^-\overleftarrow{P}v_1$$

is a cycle of order $p-1$, that is $c \geq p-1$.

Case 2.2.2. $N^-(v_1) \cap N^+(v_p) = \emptyset$.

Since $y_f \prec x_t$, we can choose two integers $1 \leq a \leq t$ and $1 \leq b \leq f$ such that $y_b \prec x_a$, and $|V(y_b\overrightarrow{P}x_p)|$ is minimum. Put

$$C = v_1x_a\overrightarrow{P}v_py_b\overleftarrow{P}v_1.$$

Clearly,

$$(N(v_1) \cup N^+(v_p)) - y_b^+ \subseteq V(C).$$

Hence,

$$c \geq |V(C)| \geq |(N(v_1) \cup N^+(v_p)) - y_b^+| + |\{v_1\}|$$
$$= |N(v_1)| + |N^+(v_p)| = |N(v_1)| + |N(v_p)| = d(v_1) + d(v_p).$$

By the hypothesis, the following vertices

$$x_1^-, x_2^-, ..., x_t^-, y_1^+, y_2^+, ..., y_f^+$$

are pairwise distinct. By $(a1)$ and $(a2)$,

$$d(v_1) = \max\{d(x_1^-), d(x_2^-), ..., d(x_t^-), d(y_1^+), d(y_2^+), ..., d(y_f^+)\}$$

$$\geq \max\{d_1, d_2, ..., d_{t+f}\} = d_{t+f} = d_{d(v_1)+d(v_p)} \geq d_{\sigma_2},$$
$$d(v_p) = \max\{d(y_1^+), d(y_2^+), ..., d(y_f^+)\}$$
$$\geq \max\{d_1, d_2, ..., d_f\} = d_f = d_{d(v_p)} \geq d_\delta,$$

implying that

$$c \geq d(v_1) + d(v_p) \geq d_\delta + d_{\sigma_2}. \qquad \blacksquare$$

References

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Submitted 22.08.2017, accepted 14.12.2017.

Աստիճանային հաջորդականություններ և դոմինանտ ցիկլեր 2-կապակցված գրաֆներում

Մ. Քուլաքզյան

Ամփոփում

Դիցուք G-ն n գագաթանի գրաֆ է δ նվազագույն աստիճանով և $\delta = d_1 \le d_2 \le \ \le d_n$ աստիճանային հաջորդականությամբ: Ամենաերկար ցիկլի երկարությունը նշանակվում է c-ով, իսկ ամենաերկար շղթայի երկարությունը (գագաթների քանակը) p-ով: 1952-ին Դիրակն ապացուցեց. (1) եթե G-ն 2-կապակցված գրաֆ է, ապա $c \ge \min\{n, 2d_1\}$, (2) կամայական գրաֆ, որը բավարարում է $d_1 \ge \frac{n}{2}$ պայմանին, համիլտոնյան է: Վերջերս այս արդյունքները լավացվեցին աստիճանային հաջորդականությունների լեզվով՝ (3) կամայական 2-կապակցված գրաֆում $c \ge \min\{n, d_\delta + d_{\delta+1}\}$, (4) $d_\delta + d_{\delta+1} \ge n$ պայմանին բավարարող կամայական գրաֆ համիլտոնյան է (Zh.G. Nikoghosyan, Degree Sequences and Long Cycles in Graphs, ArXiv:1711.04134): Ներկա աշխատանքում բերվում են (3) և (4) թեորեմների տարբերակները դոմինանտ ցիկլերի համար: Ստացված արդյունքները լավացնելի չեն:

Степенные последовательности и доминантные циклы в 2-связных графах

М. Кулакзян

Аннотация

Пусть G является n вершинным графом с минимальной степенью δ и степенной последовательностью $\delta = d_1 \le d_2 \le \quad \le d_n$. Длина длиннейшего цикла обозначается через c, а длина длиннейшей цепи (число её вершин) - через p. В 1952 году Дирак доказал: (1) если G является 2-связным графом, то $c \ge \min\{n, 2d_1\}$; (2) если граф удовлетворяет условию $d_1 \ge \frac{n}{2}$, то он является гамильтоновым. Недавно эти результаты были улучшены в терминах степенных последовательностей: (3) если G является 2-связным графом, то $c \ge \min\{n, d_\delta + d_{\delta+1}\}$; (4) если граф удовлетворяет условию $d_\delta + d_{\delta+1} \ge n$, то он является гамильтоновым (Zh.G. Nikoghosyan, Degree Sequences and Long Cycles in Graphs, ArXiv:1711.04134). В настоящей работе представляются версии теорем (3) и (4) для доминантных циклов. Полученные результаты неулучшаемы.