Mathematical Problems of Computer Science 55, 9–25, 2021.
doi: 10.51408/1963-0069

UDC 519.1

A Theorem on Even Pancyclic Bipartite Digraphs

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Abstract

We prove a Meyniel-type condition and a Bang-Jensen, Gutin and Li-type condition for a strongly connected balanced bipartite digraph to be even pancyclic.

Let D be a balanced bipartite digraph of order $2a \geq 6$. We prove that

(i) If $d(x)+d(y) \geq 3a$ for every pair of vertices x, y from the same partite set, then D contains cycles of all even lengths $2, 4, \ldots, 2a$, in particular, D is Hamiltonian.

(ii) If D is other than a directed cycle of length $2a$ and $d(x)+d(y) \geq 3a$ for every pair of vertices x, y with a common out-neighbor or in-neighbor, then either D contains cycles of all even lengths $2, 4, \ldots, 2a$ or $d(u)+d(v) \geq 3a$ for every pair of vertices u, v from the same partite set. Moreover, by (i), D contains cycles of all even lengths $2, 4, \ldots, 2a$, in particular, D is Hamiltonian.

Keywords: Digraphs, Hamiltonian cycles, Bipartite digraphs, Pancyclic, Even pancyclic.

Article info: Received 3 December 2020; accepted 4 March 2021.

1. Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1]. Every cycle and path is assumed simple and directed. A cycle in a digraph D is called *Hamiltonian* if it includes all the vertices of D. A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle. A digraph D of order $n \geq 3$ is *pancyclic* if it contains cycles of every length k, $3 \leq k \leq n$.

There are numerous sufficient conditions for the existence of a Hamiltonian cycle in a digraph (see, e.g., [1] - [10]). It was proved (see, e.g., [1], [6], [8], [9], [11] - [14]) that a number of sufficient conditions for a digraph (undirected graph) to be Hamiltonian are also sufficient for the digraph to be pancyclic (with some exceptions). For hamiltonicity, the more general and classical one is the following theorem due to M. Meyniel.

Theorem 1: (Meyniel [10]). *Let D be a strong digraph of order $n \geq 2$. If $d(x)+d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

Notice that Meyniel's theorem is a generalization of Ghouila-Houri's and Woodall's theorems.

A digraph D is a *bipartite* if there exists a partition X, Y of its vertex set into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called *balanced* if $|X| = |Y|$. Following [1], we will say that a balanced bipartite digraph D of order $2a$ is *even pancyclic* (note that a number of authors use the term *"bipancyclic"* instead of *"even pancyclic"*) if it contains cycles of all even lengths $4, 6, \dots, 2a$.

An analogue of Meyniel's theorem for the hamiltonicity of balanced bipartite digraphs was given by Adamus et al. [3].

Theorem 2: (Adamus et al. [3]). *Let D be a balanced bipartite digraph of order $2a \geq 4$. Then D is Hamiltonian provided one of the following holds:*

(a) $d(x) + d(y) \geq 3a + 1$ for each pair of non-adjacent vertices $x, y \in V(D)$;

(b) D is strong and $d(x) + d(y) \geq 3a$ for each pair of non-adjacent vertices $x, y \in V(D)$;

(c) the minimal degree of D is at least $(3a + 1)/2$;

(d) D is strong, and the minimal degree of D is at least $3a/2$.

Meszka [15] investigated the even pancyclicity of a balanced bipartite digraph satisfying a weaker condition than those in Theorem 2(a). He proved the following theorem.

Theorem 3: (Meszka [15]). *Let D be a balanced bipartite digraph of order $2a \geq 4$. Suppose that $d(x) + d(y) \geq 3a + 1$ for each two distinct vertices x, y from the same partite set. Then D contains cycles of all even lengths $4, 6, \dots, 2a$.*

Let x, y be a pair of distinct vertices in a digraph D. The pair $\{x, y\}$ is a *dominated pair* (respectively, *dominating pair*) if there is a vertex $z \in V(D) \setminus \{x, y\}$ such that $z \to \{x, y\}$ (respectively, $\{x, y\} \to z$). We will say that a pair of vertices $\{u, v\}$ is a *good* pair if it is dominated or dominating. In this case we will say that u (respectively, v) is a partner of v (respectively, u). In [5], Bang-Jensen et al. gave a new type condition for a digraph to be Hamiltonian. In the same paper, they also conjectured the following strengthening of Meyniel's theorem.

Conjecture 1: *Let D be a strong digraph of order n. Suppose that $d(x) + d(y) \geq 2n - 1$ for every good pair of non-adjacent distinct vertices x, y. Then D is Hamiltonian.*

They also conjectured that this can even be generalized to the following:

Conjecture 2:. *Let D be a strong digraph of order n. Suppose that $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent distinct vertices x, y with a common in-neighbor. Then D is Hamiltonian.*

In [5] and [4], it was proved that Conjecture 1 (2) is true if we also require an additional condition.

Theorem 4: (Bang-Jensen et al. [5]). *Let D be a strong digraph of order $n \geq 2$. Suppose that $min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbor. Then D is Hamiltonian.*

In [4], it was proved that if in Conjecture 1 we replace the degree condition $d(x)+d(y) \geq 2n-1$ with $d(x)+d(y) \geq 5n/2-4$, then Conjecture 1 is true.

There are some versions of Conjecture 1 and 2 for balanced bipartite digraphs. (see, e.g., Theorems 5, 6 and 7).

Theorem 5: (Adamus [2]). *Let D be a strong balanced bipartite digraph of order $2a \geq 6$. If $d(x)+d(y) \geq 3a$ for every good pair of distinct vertices x, y, then D is Hamiltonian.*

An analogue of Theorem 4 was given by Wang [16], and recently strengthened by the author [17].

Theorem 6: (Wang [16]). *Let D be a strong balanced bipartite digraph of order $2a \geq 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ and $d(y) \geq a+1$ or $d(y) \geq 2a-1$ and $d(x) \geq a+1$. Then D is Hamiltonian.*

Before stating the next theorem we need to define a digraph of order eight.

Example 1: *Let $D(8)$ be the bipartite digraph with partite sets $X=\{x_0, x_1, x_2, x_3\}$ and $Y=\{y_0, y_1, y_2, y_3\}$, and $A(D(8))$ contains exactly the arcs y_0x_1, y_1x_0, x_2y_3, x_3y_2 and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i$, $i \in [0,3]$, $y_0 \leftrightarrow x_2$, $y_0 \leftrightarrow x_3$, $y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$.*

It is not difficult to check that $D(8)$ is strongly connected, $max\{d(x), d(y)\} \geq 2a-1$ for every pair of vertices $\{x, y\}$ with a common out-neighbor, but it is not Hamiltonian.

Indeed, if C is a Hamiltonian cycle in $D(8)$, then C would contain the arcs x_1y_1 and x_0y_0 and therefore, the path $x_1y_1x_0y_0$ or the path $x_0y_0x_1y_1$, which is impossible since $N^-(x_0)=N^-(x_1)=\{y_0, y_1\}$.

Theorem 7: (Darbinyan [17]). *Let D be a strong balanced bipartite digraph of order $2a \geq 8$. Suppose that $max\{d(x), d(y)\} \geq 2a-1$ for every pair of distinct vertices $\{x, y\}$ with a common out-neighbor. Then D is Hamiltonian unless D is isomorphic to the digraph $D(8)$.*

Motivated by the Bondy famous metaconjecture, the author, together with Karapetyan [20],proposed the following problem:

Problem 1: Characterize those digraphs, which satisfy the conditions of Theorem 5 (or 6 or 7) but are not even pancyclic.

This problem for Theorems 6 and 7 was solved by the author [18] (Theorem 8(ii)), and for Theorem 5 by Adamus [19] (Theorem 9).

Theorem 8: *Let D be a strong balanced bipartite digraph of order $2a$.*

(i). (Darbinyan [18]). *If D contains a cycle of length $2a-2$ and $max\{d(x), d(y)\} \geq 2a-2 \geq 6$ for every pair of distinct vertices $\{x, y\}$ with a common out-neighbor, then for every k, $1 \leq k \leq a-1$, D contains a cycle of length $2k$.*

(ii). (Darbinyan [18]). *If D is not a directed cycle of length $2a \geq 8$ and $max\{d(x), d(y)\} \geq 2a-1$ for every pair of distinct vertices $\{x, y\}$ with a common out-*

neighbor, then for every k, $1 \leq k \leq a$, D contains a cycle of length $2k$ (in particular, D is even pancyclic) unless D is isomorphic to the digraph $D(8)$.

(iii). (Darbinyan and Karapetyan [20]). *Suppose that D is not a directed cycle of length $2a \geq 10$ and $max\{d(x), d(y)\} \geq 2a - 2$ for every pair of distinct vertices $\{x, y\}$ with a common out-neighbor. Then D contains a cycle of length $2a - 2$ unless D is isomorphic to a digraph of order ten, which we specify.*

The following theorem by Adamus (Theorem 9) and the main result of this paper (Theorem 10) were proved simultaneously and independently.

Theorem 9: (Adamus [19]). *Let D be a balanced bipartite Hamiltonian digraph of order $2a \geq 6$ other than a directed cycle of length $2a$. Suppose that $d(x)+d(y) \geq 3a$ for every good pair of distinct vertices x, y. Then D contains cycles of all even lengths $2, 4, \dots, 2a$.*

Theorem 10: *Let D be a strong balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. If $d(x) + d(y) \geq 3a$ for every pair of distinct vertices $\{x, y\}$ either both in X or both in Y, then D contains cycles of all even lengths less than or equal to $2a$ (in particular, D is Hamiltonian).*

The last result (Theorem 10) was presented at the "International Conference Dedicated to 90th Anniversary of Sergey Mergelyan", 20-25 May, 2018, Yerevan, Armenia.

Using some arguments of [2] by Adamus, we can prove the following lemma.

Lemma 1: *Let D be a balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. Suppose that D is not a directed cycle of length $2a$ and $d(u) + d(v) \geq 3a$ for every good pair of distinct vertices u, v. Then D either is even pancyclic or every pair of distinct vertices $\{x, y\}$ from the same partite set is a good pair.*

The following theorem follows from Theorem 10 and Lemma 1.

Theorem 11: *Let D be a strong balanced bipartite digraph of order $2a \geq 6$ other than a directed cycle of length $2a$. Suppose that $d(x) + d(y) \geq 3a$ for every good pair of distinct vertices x, y. Then D contains cycles of all even lengths $2, 4, \dots, 2a$.*

It is worth to noting that in the proof of Theorem 10 does not use the fact that D is Hamiltonian. Thus, we have a common alternative proof for Theorems 2, 3, 5 and 9. Note that if a balanced bipartite digraph satisfies the condition of Theorem 2(a) (or Theorem 2(c)), then D is strong.

Example 2: *For any even integer $a \geq 2$ there is a non-strongly connected balanced bipartite digraph D of order $2a$ with partite sets X and Y, such that $d(x)+d(y) \geq 3a$ for every pair of distinct vertices $\{x, y\}$ either both in X or both in Y ,i.e., if D is not strong, then Theorem 10 is not true.*

To see this, we take two balanced bipartite complete digraphs both of order a (a is even) with partite sets U, V and Z, W, respectively. By adding all the possible arcs from Z to V and from W to U we obtain a digraph D. It is easy to check that $d(x) + d(y) \geq 3a$ for every pair of non-adjacent distinct vertices $\{x, y\}$ of D, but D is not strongly connected and hence, D is not Hamiltonian.

2. Terminology and Notations

In this paper, we consider finite digraphs without loops and multiple arcs. Terminology and notations not defined here or above are consistent with [1]. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices. If $xy \in A(D)$, then we also write $x \to y$ and say that x *dominates* y or y is an *out-neighbor* of x and x is an *in-neighbor* of y. If $x \to y$ and $y \to x$ we shall use the notation $x \leftrightarrow y$ ($x \leftrightarrow y$ is called 2-*cycle*). We set $\overrightarrow{a}[x,y] = 1$ if $xy \in A(D)$ and $\overrightarrow{a}[x,y] = 0$ if $xy \notin A(D)$.

If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. Similarly, $A \leftrightarrow B$ means that $A \to B$ and $B \to A$. If $x \in V(D)$ and $A = \{x\}$ we sometimes write x instead of $\{x\}$. Let $N_D^+(x)$, $N_D^-(x)$ denote the set of out-neighbors, respectively the set of in-neighbors of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N_D^+(x,A) = A \cap N_D^+(x)$ and $N_D^-(x,A) = A \cap N_D^-(x)$. The *out-degree* of x is $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ is the *in-degree* of x. Similarly, $d_D^+(x,A) = |N_D^+(x,A)|$ and $d_D^-(x,A) = |N_D^-(x,A)|$. The *degree* of the vertex x in D is defined as $d_D(x) = d_D^+(x) + d_D^-(x)$ (similarly, $d_D(x,A) = d_D^+(x,A) + d_D^-(x,A)$). We omit the subscript if the digraph is clear from the context. The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D[A]$.

For integers a and b, $a \le b$, let $[a,b]$ denote the set of all the integers, which are not less than a and are not greater than b.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \ge 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). The *length* of a cycle or a path is the number of its arcs. We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-*path.* If a digraph D contains a path from a vertex x to a vertex y we say that y is *reachable* from x in D. In particular, x is reachable from itself.

We denote by $K_{a,b}^*$ the complete bipartite digraph with partite sets of cardinalities a and b. A digraph D is *strongly connected* (or, just, *strong*) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. Two distinct vertices x and y are *adjacent* if $xy \in A(D)$ or $yx \in A(D)$ (or both).

Let D be a bipartite digraph with partite sets X and Y. A *matching* from X to Y (from Y to X) is an independent set of arcs with origin in X and terminus in Y (origin in Y and terminus in X). (A set of arcs with no common end-vertices is called *independent*). If D is balanced, one says that such a matching is *perfect* if it consists of precisely $|X|$ arcs.

3. Preliminaries

In [21] and [11], the author studied pancyclicity of a digraph with the condition of the Meyniel theorem. Before stating the main result of [11] we need to define a family of digraphs.

Definition 1: *For any integers n and m, $(n+1)/2 < m \le n-1$, let Φ_n^m denote the set of digraphs D, which satisfy the following conditions: (i) $V(D) = \{x_1, x_2, \ldots, x_n\}$; (ii)*

$x_nx_{n-1}\ldots x_2x_1x_n$ is a Hamiltonian cycle in D; (iii) for each k, $1 \le k \le n-m+1$, the vertices x_k and x_{k+m-1} are not adjacent; (iv) $x_jx_i \notin A(D)$ whenever $2 \le i+1 < j \le n$ and (v) the sum of degrees for any two distinct non-adjacent vertices is at least $2n-1$.

Theorem 12: (Darbinyan [11]). *Let D be a strong digraph of order $n \ge 3$. Suppose that $d(x)+d(y) \ge 2n-1$ for all pairs of distinct non-adjacent vertices x, y in D. Then either (a) D is pancyclic or (b) n is even and D is isomorphic to one of digraphs $K^*_{n/2,n/2}$, $K^*_{n/2,n/2} \setminus \{e\}$, where e is an arbitrary arc of $K^*_{n/2,n/2}$, or (c) $D \in \Phi^m_n$ (in this case D does not contain only a cycle of length m).*

Later, Theorem 12, was also proved independently by Benhocine [22].

Lemma 2: (Adamus et al. [3]). *Let D be a strong balanced bipartite digraph of order $2a \ge 4$ with partite sets X and Y. If $d(x)+d(y) \ge 3a$ for every pair of distinct vertices x, y from the same partite set, then D contains a perfect matching from Y to X and a perfect matching from X to Y.*

Following [15], we give the following definition.

Definition 2: *Let D be a balanced bipartite digraph of order $2a \ge 4$ with partite sets X and Y. Let $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1,2,\ldots,a\}$ be a perfect matching from Y to X. We define a digraph $D^*[M_{y,x}]$ with vertex set $\{v_1, v_2, \ldots, v_a\}$ as follows: each vertex v_i corresponds to a pair $\{x_i, y_i\}$ of vertices in D and for each pair of distinct vertices v_l, v_j, $v_lv_j \in A(D^*[M_{y,x}])$ if and only if $x_ly_j \in A(D)$.*

Let D be a balanced bipartite digraph with partite sets X and Y. Let $M_{y,x}$ be a perfect matching from Y to X in D and $D^*[M_{y,x}]$ be its corresponding digraph. Further, in this paper, we will denote the vertices of D (respectively, of $D^*[M_{y,x}]$) by letters x, y (respectively, u, v) with subscripts or without them.

The *size* of a perfect matching $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1,2,\ldots,a\}$ from Y to X in D (denoted by $s(M_{y,x})$) is the number of arcs y_ix_i such that $x_iy_i \notin A(D)$.

Using the arguments of [15] by Meszka, we can formulate the following lemma.

Lemma 3: *Let D be a balanced bipartite digraph of order $2a \ge 6$ with partite sets X and Y. Let $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1,2,\ldots,a\}$ be a perfect matching from Y to X. Then the following hold:*

(i). $d^+(v_i) = d^+(x_i) - \overrightarrow{a}[x_i,y_i]$ and $d^-(v_i) = d^-(y_i) - \overrightarrow{a}[x_i,y_i]$.

(ii). If $D^*[M_{y,x}]$ contains a cycle of length k, where $k \in [2,a]$, then D contains a cycle of length $2k$.

(iii). Suppose that a is even, and $D^*[M_{y,x}]$ is isomorphic to $K^*_{a/2,a/2}$ with partite sets $\{v_1, v_2, \ldots, v_{a/2}\}$ and $\{v_{a/2+1}, v_{a/2+2}, \ldots, v_a\}$. If D contains an arc from $\{y_1, y_2, \ldots, y_{a/2}\}$ to $\{x_{a/2+1}, x_{a/2+2}, \ldots, x_a\}$, say $y_{a/2}x_a \in A(D)$, then D contains a cycle of length $2k$ for all $k = 2,3,\ldots,a$.

Proof. The proof of Lemma 3 can be found in [15], but we give it here for completeness. (i). It follows immediately from the definition of $D^*[M_{y,x}]$.

(ii). Indeed, if $v_{i_1}v_{i_2}\ldots v_{i_k}v_{i_1}$ is a cycle of length k in $D^*[M_{y,x}]$, then $y_{i_1}x_{i_1}y_{i_2}x_{i_2}y_{i_3}\ldots$

$y_{i_k}x_{i_k}y_{i_1}$ is a cycle of length $2k$ in D.

(iii). By (ii), it is clear that D contains cycles of every length $4k$, $k = 1, 2, \ldots, a/2$. It remains to show that D also contains cycles of every length $4k+2$, $k = 1, 2, \ldots, a/2-1$. Indeed, since $x_iy_j \in A(D)$ and $x_jy_i \in A(D)$ for all $i \in [1, a/2]$, $j \in [a/2+1, a]$ and $y_{a/2}x_a \in A(D)$, from the definition of $D^*[M_{y,x}]$ it follows that $y_1x_1y_{a/2+1}x_{a/2+1}y_2x_2y_{a/2+2}x_{a/2+2}y_3$ $x_3 \ldots x_ky_{a/2+k}x_{a/2+k}y_{a/2}x_ay_1$ is a cycle of length $4k+2$ in D. □

Lemma 4: (Adamus [2]). Let D be a balanced bipartite digraph of order $2a \geq 6$ other than a directed cycle of length $2a$. Suppose that $d(x) + d(y) \geq 3a$ for every good pair $\{x, y\}$ of distinct vertices in D. Then $d(u) \geq a$ for all $u \in V(D)$.

Now let us prove Lemma 1. For convenience, we will restate it here.

Lemma 1: *Let D be a balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. Suppose that D is not a directed cycle of length $2a$ and $d(u) + d(v) \geq 3a$ for every good pair of distinct vertices u, v. Then D either is even pancyclic or every pair of distinct vertices $\{x, y\}$ from the same partite set is a good pair.*

Proof: Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_a\}$. Suppose that $V(D)$ contains a pair of vertices from the same partite set, which is not a good pair. Without loss of generality, assume that $\{x_1, x_2\}$ is not a good pair. Then

$$N^+(x_1) \cap N^+(x_2) = N^-(x_1) \cap N^-(x_2) = \emptyset, d^+(x_1) + d^+(x_2) \leq a\ , d^-(x_1) + d^-(x_2) \leq a.$$

Hence, $d(x_1) + d(x_2) \leq 2a$. This together with $d(x_1) \geq a$ and $d(x_2) \geq a$ (Lemma 4) implies that $d(x_1) = d(x_2) = d^+(x_1) + d^+(x_2) = d^-(x_1) + d^-(x_2) = a$. Now we obtain that $N^+(x_1) \cup N^+(x_2) = N^-(x_1) \cup N^-(x_2) = Y$.

Let $x_i \in X \setminus \{x_1, x_2\}$ be an arbitrary vertex. We claim that $\{x_1, x_i\}$ or $\{x_2, x_i\}$ is a good pair. Assume that this is not the case. Then $(N^+(x_1) \cup N^+(x_2)) \cap N^+(x_i) = \emptyset$, which contradicts the facts that D is strong and $N^+(x_1) \cup N^+(x_2) = Y$. Thus, $\{x_1, x_i\}$ or $\{x_2, x_i\}$ is a good pair for all i, $3 \leq i \leq a$. Therefore, from condition (A) and $d(x_1) = d(x_2) = a$ it follows that $d(x_i) = 2a$ for all i, $3 \leq i \leq a$, i.e., $D[X \cup Y \setminus \{x_1, x_2\}]$ is a complete bipartite digraph with partite sets $X \setminus \{x_1, x_2\}$ and Y.

From $d(x_3) = 2a$ it follows that $d^+(x_3) = d^-(x_3) = a$. Therefore, if D contains a Hamiltonian cycle, then D contains cycles of all even lengths $2, 4, \ldots, 2a$.

Now we will show that D contains a Hamiltonian cycle.

Assume first that there is an (x_1, x_2)-path of length two. Let $x_1y_1x_2$ be an (x_1, x_2)-path of length two. Then $y_1x_1 \notin A(D)$ and $x_2y_1 \notin A(D)$ as $\{x_1, x_2\}$ is not a good pair. Now, since $x_2y_1 \notin A(D)$ and $d^+(x_2) \geq 1$, we may assume that $x_2y_2 \in A(D)$. From $d^-(x_1)+d^-(x_2) = a \geq 3$ it follows that $d^-(x_1) \geq 2$ or $d^-(x_2) \geq 2$. Assume that $d^-(x_1, \{y_3, y_4, \ldots, y_a\}) \geq 1$. We may assume that $y_3x_1 \in A(D)$. Now using the fact that $D[X \cup Y \setminus \{x_1, x_2\}]$ is a complete bipartite digraph, we see that $y_3x_1y_1x_2y_2x_3y_4x_4 \ldots y_ax_ay_3$ is a Hamiltonian cycle in D. Assume now that $d^-(x_1, \{y_3, y_4, \ldots, y_a\}) = 0$. Then from $y_1x_1 \notin A(D)$ and $d^-(x_1) = 1$ it follows that $y_2x_1 \in A(D)$. Then $x_2y_2x_1$ is an (x_2, x_1)-path of length two and $d^-(x_2) \geq 2$. Now, we have that $y_2x_2 \notin A(D)$ since $\{x_1, x_2\}$ is not a good pair. Therefore, $d^-(x_2, \{y_3, y_4, \ldots, y_a\}) \geq 1$. Now, by repeating the above argument, we conclude that D is Hamiltonian. Similarly, one can show that if there is an (x_2, x_1)-path of length two, then again D is Hamiltonian.

Assume next that there is no path of length two between x_1 and x_2. Then $d^-(x_1, N^+(x_2)) = d^-(x_2, N^+(x_1)) = 0$, and from $N^-(x_1) \cup N^-(x_2) = Y$ it follows that $N^-(x_1) = N^+(x_1)$ and $N^-(x_2) = N^+(x_2)$. This together with $d(x_1) = d(x_2) = a$ implies that $|N^+(x_1)| = |N^+(x_2)| = a/2$, a is even and $a \geq 4$. Without loss of generality, we assume that $x_1 \leftrightarrow \{y_1, y_2\}$ and $x_2 \leftrightarrow \{y_{a-1}, y_a\}$. Now, since $D[X \cup Y \setminus \{x_1, x_2\}]$ is a complete bipartite digraph, it is not difficult to check that $x_3y_1x_1y_2x_4y_3x_5y_4 \ldots x_{a-1}y_{a-2}x_ay_{a-1}x_2$ y_ax_3 is a Hamiltonian cycle in D. Thus, in all possible cases, D is Hamiltonian. Lemma 1 is proved. □

4. Proof of the Main Result

Let D be a strong balanced bipartite digraph of order $2a$. We say that D satisfies condition (A) when $d(x) + d(y) \geq 3a$ for all distinct vertices x, y from the same partite set.

The proof of Theorem 10 will be based on the following three lemmas below.

Lemma 5: *Let D be a strong balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. If D satisfies condition (A), then D contains cycles of lengths 2 and 4.*

Proof: From condition (A) immediately follows that D contains a cycle of length 2. We will prove that D also contains a cycle of length 4. By Lemma 2, D contains a perfect matching from Y to X. Let $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1, 2, \ldots, a\}$ be a perfect matching from Y to X. If for some integers i, j, $1 \leq i \neq j \leq a$, the arcs x_iy_j, x_jy_i are in D, then $x_iy_jx_jy_ix_i$ is a cycle of length 4. We may, therefore, assume that for every pair of integers i, j, $1 \leq i \neq j \leq a$, $\overrightarrow{a}[x_i, y_j] + \overrightarrow{a}[x_j, y_i] \leq 1$. Therefore, for all $i \in [1, a]$,

$$d^-(y_i) \leq a - d^+(x_i) - 1, \text{ if } \overrightarrow{a}[x_i, y_i] = 0 \text{ and } d^-(y_i) \leq a - d^+(x_i) + 1, \text{ if } \overrightarrow{a}[x_i, y_i] = 1. \quad (1)$$

Assume that there are two distinct integers i, j, $1 \leq i, j \leq a$, such that $\overrightarrow{a}[x_i, y_i] = \overrightarrow{a}[x_j, y_j] = 0$. Then, by (1), $d^-(y_i) + d^+(x_i) \leq a - 1$ and $d^-(y_j) + d^+(x_j) \leq a - 1$. These together with condition (A) and the fact that the semi-degrees of every vertex in D are bounded above by a thus implies that

$$6a \leq d(x_i) + d(x_j) + d(y_i) + d(y_j) = d^-(y_i) + d^+(x_i) + d^-(y_j) + d^+(x_j)$$
$$+ d^+(y_i) + d^+(y_j) + d^-(x_i) + d^-(x_j) \leq 6a - 2,$$

which is a contradiction.

Assume now that for some $i \in [1, a]$, $\overrightarrow{a}[x_i, y_i] = 0$ and for all $j \in [1, a] \setminus \{i\}$, $\overrightarrow{a}[x_j, y_j] = 1$. Without loss of generality, we may assume that $i = 1$. By (1), $d^-(y_1) + d^+(x_1) \leq a - 1$ and $d^-(y_2) + d^+(x_2) \leq a + 1$. If for some $k \in [3, a]$, $y_2x_k \in A(D)$ and $y_kx_2 \in A(D)$, then $x_2y_2x_ky_kx_2$ is a cycle of length 4 in D. We may, therefore, assume that $\overrightarrow{a}[y_2, x_k] + \overrightarrow{a}[y_k, x_2] \leq 1$ for all $k \in [3, a]$. This implies that

$$d^-(x_2) + d^+(y_2) = d^-(x_2, \{y_1, y_2\}) + d^+(y_2, \{x_1, x_2\}) + d^-(x_2, Y \setminus \{y_1, y_2\})$$
$$+ d^+(y_2, X \setminus \{x_1, x_2\}) \leq 4 + a - 2 = a + 2.$$

Using the above inequalities and condition (A), we obtain

$$6a \leq d(x_1) + d(x_2) + d(y_1) + d(y_2) = d^-(y_1) + d^+(x_1) + d^-(y_2) + d^+(x_2)$$

$$+d^-(x_2)+d^+(y_2)+d^-(x_1)+d^+(y_1)\leq 5a+2,$$

which is a contradiction since $a\geq 3$.

Assume finally that $x_iy_i\in A(D)$ for all $i\in[1,a]$. In this case, by the symmetry between the vertices x_i and y_i, similar to (1), we obtain that $d^-(x_i)+d^+(y_i)\leq a+1$. This together with (1) implies that for any i, j $(1\leq i\neq j\leq a)$,

$$6a\leq d(x_i)+d(x_j)+d(y_i)+d(y_j)\leq 4a+4,$$

a contradiction since $a\geq 3$. Lemma 5 is proved. □

Remark 1: There is a strong balanced bipartite digraph of order 4, which satisfies condition (A), but contains no cycle of length 4. To see this, we consider the following digraph with vertex set $V(D)=\{x_1,x_2,y_1,y_2\}$ and arc set $D(A)=\{x_1y_2,y_2x_2,x_2y_2,x_2y_1,$
$y_1x_1\}$.

Lemma 6: *Let D be a strong balanced bipartite digraph of order $2a\geq 6$ with partite sets X and Y. Let $M_{y,x}=\{y_ix_i\in A(D)\,|\,i=1,2,\dots,a\}$ be a perfect matching from Y to X in D such that the size $s(M_{y,x})$ of $M_{y,x}$ is maximum among the sizes of all the perfect matching from Y to X in D. If D satisfies condition (A), then the digraph $D^*[M_{y,x}]$ either is strong or D contains cycles of all lengths $2,4,\dots,2a$.*

Proof: Notice that, by Lemma 5, D contains cycles of lengths 2 and 4. Suppose that the digraph $D^*[M_{y,x}]$ is not strong. Then in $D^*[M_{y,x}]$ there are two distinct vertices, say v_1 and v_j, such that there is no path from v_1 to v_j in $D^*[M_{y,x}]$. Let U be the set of all vertices reachable from v_1 and W be the set of all vertices from which v_j is reachable. Notice that $v_1\in U$, $v_j\in W$ and $U\cap W=\emptyset$.

Case 1. $d^+(v_1)\geq 1$ and $d^-(v_j)\geq 1$.

Then $|U|\geq 2$ and $|W|\geq 2$. Let v_l, v_k be two distinct vertices in U and v_p, v_q be two distinct vertices in W. From condition (A) and the fact that the semi-degrees of every vertex in D are bounded above by a it follows that

$$d^+(x_l)+d^+(x_k)\geq a\quad\text{and}\quad d^-(x_p)+d^-(x_q)\geq a. \tag{2}$$

By Lemma 3(i),

$$d^+(v_l)+d^+(v_k)=d^+(x_l)+d^+(x_k)-\overrightarrow{a}[x_l,y_l]-\overrightarrow{a}[x_k,y_k],$$

and

$$d^-(v_p)+d^-(v_q)=d^-(y_p)+d^-(y_q)-\overrightarrow{a}[x_p,y_p]-\overrightarrow{a}[x_q,y_q]. \tag{3}$$

It follows from them and (2) that $d^+(v_l)+d^+(v_k)\geq a-2$ and $d^-(v_p)+d^-(v_q)\geq a-2$. Without loss of generality we may assume that $d^+(v_l)\geq (d^+(v_l)+d^+(v_k))/2$ and $d^-(v_p)\geq (d^-(v_p)+d^-(v_p))/2$. These imply that $d^+(v_l)\geq (a-2)/2$ and $d^-(v_p)\geq (a-2)/2$, which in turn imply that $|U|\geq a/2$ and $|W|\geq a/2$.

If $d^+(v_l)+d^+(v_k)\geq a-1$ or $d^-(v_p)+d^-(v_q)\geq a-1$, then $|U|\geq (a+1)/2$ or $|W|\geq (a+1)/2$, respectively. Hence $|U|+|W|\geq (2a+1)/2$, which is a contradiction since $|U|+|W|\leq a$. Using (2) and (3), we may therefore assume that

$$d^+(v_l)+d^+(v_k)=d^+(x_l)+d^+(x_k)-2=d^-(v_p)+d^-(v_q)=d^-(y_p)+d^-(y_q)-2=a-2.$$

Then it is easy to see that the arcs x_ly_l, x_ky_k, x_py_p and x_qy_q are in D, $|U| = |W| = a/2$ and $V(D^*[M_{y,x}]) = U \cup W$. In particular, a is even. Without loss of generality, we assume that $U = \{v_1, v_2, \ldots, v_{a/2}\}$ and $W = \{v_{a/2+1}, v_{a/2+2}, \ldots, v_a\}$. Since there is no arc from a vertex in U to a vertex in W, the following holds:

$$A(\{x_1, x_2, \ldots, x_{a/2}\} \to \{y_{a/2+1}, y_{a/2+2}, \ldots, y_a\}) = \emptyset. \tag{4}$$

Therefore, if $i \in [1, a/2]$ and $j \in [a/2+1, a]$, then $d^+(x_i) \leq a/2$ and $d^-(y_j) \leq a/2$. Together with (2) they imply that $d^+(x_i) = d^-(y_j) = a/2$ and

$$x_i \to \{y_1, y_2, \ldots, y_{a/2}\} \quad \text{and} \quad \{x_{a/2+1}, x_{a/2+2}, \ldots, x_a\} \to y_j \tag{5}$$

for all $i \in [1, a/2]$ and $j \in [a/2+1, a]$, respectively. Therefore, by condition (A),

$$3a \leq d(x_i) + d(x_k) \leq a + d^-(x_i) + d^-(x_k),$$

for every pair of $i, k \in [1, a/2]$. This implies that $d^-(x_i) = d^-(x_k) = a$, which means that $\{y_1, y_2, \ldots, y_a\} \to \{x_i, x_k\}$. Similarly, $y_j \to \{x_1, x_2 \ldots, x_a\}$, for all $j \in [a/2+1, a]$. From this and (5) it follows that the induced subdigraphs $D[\{x_1, x_2, \ldots, x_{a/2}, y_1, y_2, \ldots, y_{a/2}\}]$ and $D[\{x_{a/2+1}, x_{a/2+2}, \ldots, x_a, y_{a/2+1}, y_{a/2+2}, \ldots, y_a\}]$ both are balanced bipartite complete digraphs. Therefore, D contains cycles of all lengths $2, 4, \ldots, a$. It remains to show that D also contains cycles of every length $a + 2b$, $b \in [1, a/2]$. Since D is strong and (4), it follows that there is an arc from a vertex in $\{y_1, y_2, \ldots, y_{a/2}\}$ to a vertex in $\{x_{a/2+1}, x_{a/2+2}, \ldots, x_a\}$. Without loss of generality, we may assume that $y_{a/2}x_{a/2+1} \in A(D)$. Then $x_1y_1x_2y_2 \ldots x_{a/2}y_{a/2}x_{a/2+1}\ y_{a/2+1}x_{a/2+2} \ldots x_{a/2+b}y_{a/2+b}x_1$ is a cycle of length $a + 2b$. Thus, D contains cycles of all lengths $2, 4, \ldots, 2a$. This completes the discussion of Case 1.

Case 2. $d^+(v_1) = 0$.

Then $d^+(x_1) = 1$ and $x_1y_1 \in A(D)$, since D is strong. Hence $d(x_1) \leq a + 1$. Together with condition A this implies that $a \leq d(x_1) \leq a + 1$. We distinguish two subcases depending on $d(x_1)$.

Case 2.1. $d(x_1) = a$.

Then $d(x_i) \geq 2a$ for all $i \in [2, a]$ because of condition A. Therefore, the induced subdigraph $D\langle Y \cup X \setminus \{x_1\}\rangle$ is a complete bipartite digraph with partite sets Y and $X \setminus \{x_1\}$. It is clear that D contains cycles of every lengths $2, 4, \ldots, 2a-2$. Since $d(x_1) = a$, $d^+(x_1) = 1$ and $a \geq 3$, we have that $d^-(x_1) = a - 1 \geq 2$. Without loss of generality we may assume that $y_2x_1 \in A(D)$. Then $y_2x_1y_1x_3y_3 \ldots x_ay_ax_2y_2$ is a cycle of length $2a$.

Case 2.2. $d(x_1) = a + 1$.

Then $\{y_1, y_2, \ldots, y_a\} \to x_1$ because of $d^+(x_1) = 1$, and, by condition (A), $d(x_i) \geq 2a - 1$ for all $i \in [2, a]$. Observe that if for some $i \in [2, a]$, $y_1x_i \in A(D)$, then $M^i_{y,x} := \{y_ix_1, y_1x_i\} \cup \{y_jx_j \,|\, j \in [1, a] \setminus \{1, i\}\}$ is a perfect matching from Y to X in D.

Assume that for some $i \in [2, a]$, $x_iy_1 \notin A(D)$. Then $y_1x_i \in A(D)$ because of $d(x_i) \geq 2a - 1$. Since $x_1y_1 \in A(D)$, $x_iy_1 \notin A(D)$ and $x_1y_i \notin A(D)$, it follows that $s(M^i_{y,x}) > s(M_{y,x})$, which contradicts the choice of $M_{y,x}$. We may therefore assume that $\{x_2, x_3, \ldots, x_a\} \to y_1$. If

$y_1x_i \in A(D)$ and $x_iy_i \in A(D)$, where $i \in [2, a]$, then again we have $s(M^i_{y,x}) > s(M_{y,x})$, since the arcs x_1y_1, x_iy_i are in D and $x_1y_i \notin A(D)$. We may therefore assume that $\overrightarrow{a}[y_1, x_i] + \overrightarrow{a}[x_i, y_i] \leq 1$. This together with $d(x_i) \geq 2a - 1$, $i \in [2, a]$, implies that

$$\{y_2, y_3, \ldots, y_a\} \to x_i \to \{y_2, y_3, \ldots, y_a\} \setminus \{y_i\}. \tag{6}$$

Since D is strong and $d^+(x_1, \{y_2, y_3, \ldots, y_a\}) = 0$, it follows that $d^+(y_1, \{x_2, x_3, \ldots, x_a\}) \geq 1$. Without loss of generality, we assume that $y_1x_2 \in A(D)$. Then, since $y_2x_1 \in A(D)$ and (6), $x_1y_1x_2y_3x_3 \ldots x_{k-1}y_kx_ky_2x_1$ is a cycle of length $2k$ for every $k \in [3, a]$. Lemma 6 is proved. □

Lemma 7: *Let D be a strong balanced bipartite digraph of order $2a \geq 6$ with partite sets X and Y. Let $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1, 2, \ldots, a\}$ be a perfect matching from Y to X in D such that the size $s(M_{y,x})$ of $M_{y,x}$ is maximum among the sizes of all the perfect matching from Y to X in D. If D satisfies condition (A), then either $d(u) + d(v) \geq 2a - 1$ for every pair of non-adjacent vertices u, v in $D^*[M_{y,x}]$ or D contains cycles of all lengths $2, 4, \ldots, 2a$.*

Proof: Suppose that D is not even pancyclic. Then by Lemma 6, $D^*[M_{y,x}]$ is strong. Let v_i and v_j be two arbitrary distinct vertices in $D^*[M_{y,x}]$. Write

$g(i, j) := d^+(x_i) + d^+(x_j) + d^-(y_i) + d^-(y_j)$ and $f(i, j) := d^-(x_i) + d^-(x_j) + d^+(y_i) + d^+(y_j)$.

By Lemma 3(i), we have

$$d(v_i) + d(v_j) = g(i, j) - 2\overrightarrow{a}[x_i, y_i] - 2\overrightarrow{a}[x_j, y_j]. \tag{7}$$

By condition (A), we have

$$6a \leq d(x_i) + d(x_j) + d(y_i) + d(y_j) = f(i, j) + g(i, j).$$

Hence,

$$g(i, j) \geq 2a \quad \text{and} \quad 4a \geq f(i, j) \geq 6a - g(i, j). \tag{8}$$

since the semi-degrees of every vertex of D are bounded above by a. Now we prove the following claim.

Claim 1: *Assume that the vertices v_i and v_j in $D^*[M_{y,x}]$ are not adjacent. Then the following hold:*

(i). If $x_iy_i \in A(D)$ or $x_jy_j \in A(D)$, then $\overrightarrow{a}[y_i, x_j] + \overrightarrow{a}[y_j, x_i] \leq 1$.

(ii). If $x_iy_i \notin A(D)$ or $x_jy_j \notin A(D)$, then $d(v_i) + d(v_j) \geq 2a - 1$ in $D^*[M_{y,x}]$.

Proof: Since the vertices v_i and v_j in $D^*[M_{y,x}]$ are not adjacent, it follows that $x_iy_j \notin A(D)$ and $x_jy_i \notin A(D)$.

(i). Suppose, to the contrary, that $x_iy_i \in A(D)$ or $x_jy_j \in A(D)$, but $\overrightarrow{a}[y_i, x_j] + \overrightarrow{a}[y_j, x_i] = 2$. Then $M'_{y,x} := \{y_ix_j, y_jx_i\} \cup \{y_kx_k \,|\, k \in [1, a] \setminus \{i, j\}\}$ is a new perfect matching from Y to X in D. Since $x_jy_i \notin A(D)$, $x_iy_j \notin A(D)$ and $x_iy_i \in A(D)$ or $x_jy_j \in A(D)$, it follows that $s(M'_{y,x}) > s(M_{y,x})$, which contradicts the choice of $M_{y,x}$.

(ii). If $\overrightarrow{a}[x_i, y_i] = \overrightarrow{a}[x_j, y_j] = 0$, then from (7) and $g(i, j) \geq 2a$ it follows that $d(v_i) + d(v_j) \geq 2a$ in $D^*[M_{y,x}]$. We may therefore assume that $x_iy_i \in A(D)$. Then $x_jy_j \notin A(D)$ by the assumption of Claim 1(ii). If $g(i, j) \geq 2a+1$, then, by (7), $d(v_i)+d(v_j) \geq 2a-1$.

Thus, we may assume that $g(i,j) = 2a$. Then $f(i,j) \geq 4a$ by (8). The last inequality implies that the arcs y_ix_j, y_jx_i are in D. Therefore, $M'_{y,x} := \{y_ix_j, y_jx_i\} \cup \{y_kx_k \,|\, k \in [1,a] \setminus \{i,j\}\}$ is a new perfect matching from Y to X in D. Since $x_iy_i \in A(D)$, $x_iy_j \notin A(D)$ and $x_jy_i \notin A(D)$, it follows that $s(M'_{y,x}) > s(M_{y,x})$, which contradicts the choice of $M_{y,x}$. The claim is proved. □

We now return to the proof of Lemma 7. Suppose that there exist two distinct non-adjacent vertices, say v_1 and v_2, in $D^*[M_{y,x}]$ such that

$$d(v_1) + d(v_2) \leq 2a - 2. \tag{9}$$

This together with (7), $\overrightarrow{a}[x_1, y_1] \leq 1$ and $\overrightarrow{a}[x_2, y_2] \leq 1$ implies that $g(1,2) \leq 2a + 2$. Therefore, $2a \leq g(1,2) \leq 2a + 2$.

Case 1. $\overrightarrow{a}[x_1, y_1] = 0$.

Then from (7), (9) and the fact that $g(1,2) \geq 2a$, it follows that $\overrightarrow{a}[x_2, y_2] = 1$ (i.e., $x_2y_2 \in A(D)$) and $g(1,2) = 2a$. From this and (8) it follows that $f(1,2) \geq 4a$, which in turn implies that $y_1x_2 \in A(D)$ and $y_2x_1 \in A(D)$. The aforementioned contradicts Claim 1(i) since $x_2y_2 \in A(D)$.

Case 2. $\overrightarrow{a}[x_1, y_1] = \overrightarrow{a}[x_2, y_2] = 1$, i.e., $x_1y_1 \in A(D)$ and $x_2y_2 \in A(D)$.

From Claim 1(i) it follows that $y_1x_2 \notin A(D)$ or $y_2x_1 \notin A(D)$. If $2a \leq g(1,2) \leq 2a+1$, then from (8) it follows that $f(1,2) \geq 4a - 1$, which in turn implies that $y_1x_2 \in A(D)$ and $y_2x_1 \in A(D)$, which is a contradiction. We may therefore assume that $g(1,2) = 2a+2$. This and (8) imply that $f(1,2) \geq 4a-2$. Then, since $y_1x_2 \notin A(D)$ or $y_2x_1 \notin A(D)$, it follows that $y_1x_2 \in A(D)$ or $y_2x_1 \in A(D)$. Without loss of generality, we may assume that $y_1x_2 \notin A(D)$ and $y_2x_1 \in A(D)$. Note that the vertices y_1 and x_2 are not adjacent. Then $f(1,2) = 4a - 2$, which in turn implies that $d^-(x_1) = d^+(y_2) = a$ and $d^-(x_2) = d^+(y_1) = a - 1$. Therefore,

$$y_2 \to \{x_1, x_2, \ldots, x_a\};\ \{y_1, y_2, \ldots, y_a\} \to x_1;\ y_1 \to \{x_1, x_3, x_4, \ldots, x_a\};$$
$$\{y_2, y_3, \ldots, y_a\} \to x_2. \tag{10}$$

since $y_1x_2 \notin A(D)$. Using (10), it is easy to see that for all $i \in [3, a]$,

$$M^i_{y,x} := \{y_2x_1, y_ix_2, y_1x_i\} \cup \{y_kx_k \,|\, k \in [3,a] \setminus \{i\}\}$$

is a perfect matching from Y to X in D. Using the facts that the arcs x_1y_1, x_2y_2 are in D, it is not difficult to see that if for some $i \in [3, a]$, either $x_2y_i \notin A(D)$ or $x_iy_1 \notin A(D)$ or $x_iy_i \in A(D)$, then $s(M^i_{y,x}) > s(M_{y,x})$, which contradicts the choice of $M_{y,x}$. We may therefore assume that $x_iy_i \notin A(D)$ for all $i \in [3, a]$, and $x_2 \to \{y_2, y_3, \ldots, y_a\}$ and $\{x_3, x_4, \ldots, x_a\} \to y_1$. Together with (10) they imply that

$$x_2 \leftrightarrow \{y_2, y_3, \ldots, y_a\} \quad \text{and} \quad y_1 \leftrightarrow \{x_1, x_3, x_4, \ldots, x_a\}. \tag{11}$$

Since the vertices y_1, x_2 are not adjacent, from (11) and Lemma 3(i) it follows that

$$d^-(y_1) = d^+(x_2) = a - 1, \quad d^-(v_1) = d^+(v_2) = a - 2. \tag{12}$$

From $g(1,2) = 2a + 2$, (7), $x_1y_1 \in A(D)$ and $x_2y_2 \in A(D)$ it follows that $d(v_1) + d(v_2) = g(1,2) - 4 = 2a - 2$. This together with (12) and the fact that $D^*[M_{y,x}]$ is strong implies that

$d^+(v_1) = d^-(v_2) = 1$. This means that $d^+(x_1) = d^-(y_2) = 2$. Therefore, $d(x_1) = d(y_2) = a+2$ by (10).

Now for every $i \in [3, a]$ we consider the perfect matching $M^i_{y,x}$ and its corresponding digraph $D^*[M^i_{y,x}]$. Notice that $s(M_{y,x}) = s(M^i_{y,x}) = a-2$, the vertices y_1, x_2 are not adjacent and the arcs x_iy_i, x_1y_2 are not in $A(D)$. Hence, the vertices $v^i_1 = \{y_1, x_i\}$, $v^i_2 = \{y_i, x_2\}$ in $D^*[M^i_{y,x}]$ are not adjacent. From Claim 1(ii) it follows that in $D^*[M^i_{y,x}]$ the degree sum of every pair of two distinct non-adjacent vertices, other than $\{v^i_1, v^i_2\}$, is at least $2a - 1$. If in $D^*[M^i_{y,x}]$, $d(v^i_1) + d(v^i_2) \leq 2a - 2$, then by the arguments to that in the proof of $d(x_1) = d(y_2) = a + 2$, we deduce that $d(x_i) = d(y_i) = a + 2$ for all $i \in [3, a]$. Therefore, for all $i \in [3, a]$, $3a \leq d(x_1) + d(x_i) \leq 2a + 4$. This means that $a \leq 4$, i.e., $a = 3$ or $a = 4$.

Let $a = 3$. By Lemma 5, it suffices to show that D contains a cycle of length 6. Using (10) and (11), it is easy to check that $x_3y_2x_2y_3x_1y_1x_3$ is a cycle of length 6 in D.

Let now $a = 4$. By Lemma 5, we need to show that D contains cycles of lengths 6 and 8. From $d(x_4) = 6$ and $x_4y_4 \notin A(D)$ it follows that $x_4y_2 \in A(D)$ or $x_4y_3 \in A(D)$.

Assume that $x_3y_4 \in A(D)$. Then using (10) and (11) it is not difficult to see that $x_3y_4x_2y_2x_1y_1x_3$ is a cycle of length 6, and $x_3y_4x_4y_2x_2y_3x_1y_1x_3$ (respectively, $x_3y_4x_4y_3x_2y_2$ $x_1y_1x_3$) is a cycle of length 8, when $x_4y_2 \in A(D)$ (respectively, when $x_4y_3 \in A(D)$).

Assume now that $x_3y_4 \notin A(D)$. Then from $x_4y_4 \notin A(D)$ and $d(y_4) = 6$ it follows that $x_1y_4 \in A(D)$. Now again using (10) and (11), we see that $x_1y_4x_2y_2x_3y_1x_1$ is a cycle of length 6, and $x_1y_4x_4y_2x_2y_3x_3y_1x_1$ (respectively, $x_1y_4x_4y_3x_2y_2x_3y_1x_1$) is a cycle length 8, when $x_4y_2 \in A(D)$ (respectively, when $x_4y_3 \in A(D)$). Thus, we have shown that if $a = 3$ or $a = 4$, then D contains cycles of all lengths $2, 4, \ldots, 2a$, which contradicts our supposition that D is not even pancyclic. This completes the proof of Lemma 7. □

We now ready to complete the proof of Theorem 10.

Proof of Theorem 10: Let D be a digraph satisfying the conditions of Theorem 10. By Lemma 5, D contains cycles of lengths 2 and 4. By Lemma 2, D contains a perfect matching from Y to X. Let $M_{y,x} = \{y_ix_i \in A(D) \,|\, i = 1, 2, \ldots, a\}$ be a perfect matching from Y to X in D with the maximum size among the sizes of all the perfect matching from Y to X in D. By Lemma 6, the digraph $D^*[M_{y,x}]$ either contains cycles of all lengths $2, 4, \ldots, 2a$ or is strongly connected. In the former case we are done. Assume that $D^*[M_{y,x}]$ is strongly connected. By Lemma 7, D either contains cycles of all lengths $2, 4, \ldots, 2a$ or (ii) $d(u) + d(v) \geq 2a - 1$ for every pair of non-adjacent vertices u, v in $D^*[M_{y,x}]$. Assume that the second case holds. Therefore, by Theorem 12, either (a) $D^*[M_{y,x}]$ contains cycles of every length k, $k \in [3, a]$ or (b) a is even and $D^*[M_{y,x}]$ is isomorphic to one of digraphs $K^*_{a/2,a/2}$, $K^*_{a/2,a/2} \setminus \{e\}$ or (c) $D^*[M_{y,x}] \in \Phi^m_a$, where $(a + 1)/2 < m \leq a - 1$.

(a). In this case, by Lemma 5 and Lemma 3(ii), D contains cycles of every length $2k$, $k \in [1, a]$.

(b). $D^*[M_{y,x}]$ is isomorphic to $K^*_{a/2,a/2}$ or $K^*_{a/2,a/2} \setminus \{e\}$ with partite sets $\{v_1, v_2, \ldots,$ $v_{a/2}\}$ and $\{v_{a/2+1}, v_{a/2+2}, \ldots, v_a\}$. Notice that $a \geq 4$ and $D^*[M_{y,x}]$ contains cycles of every length $2k$, $k \in [1, a/2]$. Therefore, by Lemma 3(ii), D contains cycles of every length $4k$, $k \in [1, a/2]$. It remains to show that for any $k \in [1, a/2 - 1]$, D also contains a cycle of length $4k + 2$.

We claim that there exist $p \in [1, a/2]$ and $q \in [a/2+1, a]$ such that $y_px_q \in A(D)$. Assume that this is not the case, i.e., there is no arc from a vertex of $\{y_1, y_2, \ldots, y_{a/2}\}$ to a vertex of $\{x_{a/2+1}, x_{a/2+2}, ldots, x_a\}$. Then, since $D^*[M_{y,x}]$ is isomorphic to $K^*_{a/2,a/2}$ or $K^*_{a/2,a/2} \setminus \{e\}$, from the definition of $D^*[M_{y,x}]$ it follows that $d^+(y_1) \leq a/2$, $d^+(y_{a/2}) \leq a/2$, $d^-(y_1) \leq a/2+1$ and $d^-(y_{a/2}) \leq a/2+1$. Combining these inequalities, we obtain that $d(y_1)+d(y_{a/2}) \leq 2a+2$, which contradicts condition (A) since $a \geq 4$.

It suffices to consider the case when $D^*[M_{y,x}]$ is isomorphic to $K^*_{a/2,a/2} \setminus \{e\}$. Without loss of generality, we may assume that $e = v_a v_{a/2}$. From the definition of $D^*[M_{y,x}]$ it follows that $\{x_1, x_2, \ldots, x_{a/2}\} \to \{y_{a/2+1}, y_{a/2+2}, \ldots, y_a\}$ and D contains all possible arcs from $\{x_{a/2+1}, x_{a/2+2}, \ldots, x_a\}$ to $\{y_1, y_2, \ldots, y_{a/2}\}$ except $x_a y_{a/2}$.

If $p = a/2$ and $q = a$ (i.e., $y_{a/2}x_a \in A(D)$), then $y_1x_1y_{a/2+1}x_{a/2+1}y_2x_2y_{a/2+2}x_{a/2+2} \cdots y_kx_ky_{a/2+k}\, x_{a/2+k}y_{a/2}x_ay_1$ is a cycle of length $4k+2$, where $k \in [1, a/2-1]$. Thus, we may assume that $y_{a/2}x_a \notin A(D)$. Then the vertices x_a, $y_{a/2}$ are not adjacent since $x_ay_{a/2} \notin A(D)$. This together with $d^-(y_{a/2}, \{x_1, x_2, \ldots, x_{a/2}\}) \leq 1$ implies that $d(y_{a/2}) \leq 3a/2 - 1$. Therefore, by condition (A), $d(y_{a/2-1}) \geq 3a/2 + 1$ and hence, $y_{a/2-1}x_a \in A(D)$ since $d^-(y_{a/2-1}, \{x_1, x_2, \ldots, x_{a/2}\}) \leq 1$. Now it is not difficult to check that if $a \geq 6$, then $y_1x_1y_{a/2+1}x_{a/2+1}y_2x_2y_{a/2+2}x_{a/2+2} \cdots y_kx_ky_{a/2+k}x_{a/2+k}y_{a/2-1}x_ay_1$ is a cycle of length $4k+2$ when $k \in [1, a/2-2]$, and $y_1x_1y_{a/2+1}x_{a/2+1}y_2x_2\, y_{a/2+2}x_{a/2+2} \cdots x_{a/2-2}y_{a-2}x_{a-2}y_{a/2}x_{a/2}\, y_{a-1}x_{a-1}y_{a/2-1}\, x_ay_1$ is a cycle of length $2a-2$. If $a = 4$, then $y_2x_2y_4x_4y_1x_3y_2$ is a cycle of length $6 = 2a-2$.

(c). $D^*[M_{y,x}] \in \Phi^m_a$. Since D contains cycles of lengths 2, 4 (Lemma 5) and every digraph in Φ^m_a is Hamiltonian, we can assume that $a \geq 4$. Let $V(D^*[M_{y,x}]) = \{v_1, v_2, \ldots, v_a\}$ and $v_av_{a-1} \ldots v_2v_1v_a$ be a Hamiltonian cycle in $D^*[M_{y,x}]$. Therefore, by the definition of $D^*[M_{y,x}]$, for all $i \in [2, a]$, $x_iy_{i-1} \in A(D)$ and $x_1y_a \in A(D)$. From the definition of Φ^m_a we have $d^+(v_a) = 1$ and $d^+(v_{a-1}) \leq 2$. This means that $d^+(x_a) \leq 2$ and $d^+(x_{a-1}) \leq 3$. These together with $d^-(x_a) \leq a$, $d^-(x_{a-1}) \leq a$ and condition (A) implies that

$$d(x_a) \leq a+2, \quad d(x_{a-1}) \leq a+3 \quad \text{and} \quad 3a \leq d(x_a) + d(x_{a-1}) \leq 2a+5. \tag{13}$$

The last inequality of (13) implies that $a \leq 5$, i.e., $a = 4$ or $a = 5$.

Let $a = 5$. Then from (13) it follows that $d(x_a) + d(x_{a-1}) = 2a+5$, $d^-(x_a) = d^-(x_{a-1}) = a$, i.e., $\{y_1, y_2, \ldots, y_a\} \to \{x_a, x_{a-1}\}$. Therefore, $y_2x_5y_4x_4y_3x_3y_2$ (respectively, $y_1x_5y_4x_4y_3x_3y_2x_2y_1$) is a cycle of length 6 (respectively, of length 8).

Let $a = 4$. In this case, we need to show that D contains a cycle of length 6. If $x_1y_3 \in A(D)$ (or $y_2x_1 \in A(D)$), then $x_1y_3x_3y_2x_2y_1x_1$ (respectively, $x_1y_4x_4y_3x_3y_2x_1$) is a cycle of length 6. We may therefore assume that $x_1y_3 \notin A(D)$ and $y_2x_1 \notin A(D)$. Then $d(x_1) = d(x_4) = 6$ since $d(x_4) \leq a+2$, $d^+(x_a) \leq 2$ and $d(x_1) + d(x_4) \geq 12$. Therefore, $d^-(x_4) = 4$, which in turn implies that $y_1x_4 \in A(D)$. Hence, $y_1x_4y_3x_3y_2x_2y_1$ is a cycle of length 6. Thus, we have shown that if $D^*[M_{y,x}] \in \Phi^m_a$, then $a = 4$ or $a = 5$ and D contains cycles of all lengths $2, 4, \ldots, 2a$. This completes the proof of the theorem. □

5. Conclusion

In the current article, we prove a Meyniel-type condition and a Bang-Jensen, Gutin and Li-type condition for a strong balanced bipartite digraph of order $2a \geq 6$ to have cycles of all even lengths less than equal to $2a$.

It is worth to noting that over the past three years, various authors have received a number of sufficient conditions for the existence of cycles with certain properties in bipartite digraphs. In particular, several sufficient conditions for a balanced bipartite digraph to be

Hamiltonian or be even pancyclic were obtained (see, e.g., [23] by Wang and Wu, [24] by Adamus, [25] by Wang, [26] by Wang et al.).

A Hamiltonian path in a digraph D in which the initial vertex dominates the terminal vertex is called a Hamiltonian bypass in D. It was proved that a number of sufficient conditions for a digraph to be Hamiltonian is also sufficient for a digraph to contain a Hamiltonian bypass with some exceptions, which are characterized in [27], and the papers cited there. It is not difficult to show that, if a balanced bipartite digraph of order $2a \geq 4$ satisfies the conditions of Theorem 2(a) (or 2(b)), then D has a Hamiltonian bypass. In this regard, we believe that the the following conjecture is true.

Conjecture 3: D be a strong balanced bipartite digraph of order $2a \geq 6$. If D satisfies the conditions one of Theorems 2, 5 and 7, then D contains a Hamiltonian bypass, with some exceptions.

To conclude this section, we mention that Wang et al. [28] constructed an infinite family of counterexamples to Conjecture 2. Note that each of these counterexamples contains a vertex, which has degree equal to three.

Thus, Conjecture 2 remains open for digraphs with the minimum degree is at least four and for k-strong digraphs, where $k \geq 2$.

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Թեորեմ երկմասանյա կողմնորոշված գրաֆների զույգ համացիկլիկության մասին

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Ամփոփում

Ներկա աշխատանքում ապացուցվել է հետևյալ թեորեմը:

Թեորեմ: Դիցուք D-ն ուժեղ կապակցված $2a \geq 6$-գագաթանի երկմասանյա հավասարակշռված կողմնորոշված գրաֆ է, իսկ $\{x, y\}$-ը D գրաֆի գագաթների ցանկացած պարտվող կամ հաղթող զույգ է: Եթե տեղի ունի $d(x) + d(y) \geq 3a$ պայմանը, ապա D -ն պարունակում $2a$-ից փոքր կամ հավասար ցանկացած զույգ երկարության ցիկլ, բացի այն դեպքից, երբ D-ն $2a$ երկարության ցիկլ է:

Բանալի բառեր՝ Կողմնորոշված գրաֆ, համիլտոնյան ցիկլ, երկմասանյա կողմնորոշված գրաֆ, համացիկլիկ, զույգ համացիկլիկ:

Теорема о четных панциклических двудольных орграфах

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Аннотация

В настоящей работе доказана следующая теорема:

Теорема: Пусть D есть сильно связный $2a \geq 6$ - вершинный балансированный двудольный орграф. Предположим, что для каждой доминирующей и каждой доминирумой пары $\{x, y\}$ различных вершин имеет место $d(x) + d(y) \geq 3a$. Тогда D содержит контур любой четной длины $2k$, $0 \leq k \leq a$, кроме случая кокда D является контуром длины $2a$.

Ключевые слова: Орграф, гамильтонов цикл, двудольний орграф, панциклический орграф, четный панциклический орграф.