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A Fixed Point Theorem for q-Lattices

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Introduction and preliminaries

B. Knaster's and A. Tarski's set-theoretical fixed point theorem is well known [1]. A generalization of this result is the lattice-theoretical fixed point theorem (named Tarski's fixed point theorem)[2]. In [3] Tarski's theorem is generalized for semilattices. In the present work a fixed point-like theorem is proved for q-lattices. The concept of a q-lattice was introduced in [4].

The algebra $(L; \cap, \cup)$ is called q-semilattice, if it satisfies the following identities: 1.$a \cap b = b \cap a$(commutativity); 2. $a \cap (b \cap c) = (a \cap b) \cap c$ (associativity); 3. $a \cap (b \cap b) = a \cap b$ (weak idempotence).

The algebra $(L; \cap, \cup)$ with two binary operations is called q-lattice, if the reducts $(L; \cap)$ and $(L; \cup)$ are q-semilattices and the following identities, $a \cap (b \cup a) = a \cap a$, $a \cup (b \cap a) = a \cup a$ (weak absorption), $a \cap a = a \cup a$ (equalization) are valid.

For example, $(Z \setminus \{0\}; \cap, \cup)$, where $x \cap y = |(x, y)|$ and $x \cup y = |[x, y]|$, for which (x, y) and $[x, y]$ are the greatest common division (gcd) and the least common multiple (lcm) of x and y , is a q-lattice, which is not a lattice, since $x \cap x \neq x$ and $x \cup x \neq x$.

The relation $Q \subseteq L \times L$ is called a quasiorder if it is reflexive and transitive. Let Q be a quasiorder on the set $L \neq \emptyset$; then $E_Q = Q \cap Q^{-1} \subseteq L \times L$ is an equivalence. The relation Q/E_Q which is induced from Q on the set L/E_Q in the following manner: $(A, B) \in Q/E_Q \leftrightarrow aQb$, $\forall a \in A, \forall b \in B$, where $A, B \in L/E_Q$, is an order. Further, the order Q/E_Q is denoted by $\leq_Q$ and the class of equivalence which includes the element x is denoted by $[x] \in L/E_Q$. The function $K : L/E_Q \to L$ is called choice function, if $K([a]) \in [a]$ for each $[a] \in L/E_Q$. The pair (L, Q) is called $infsup$-quasiordered set, if for each two classes of equivalences $[a], [b] \in L/E_Q$ there exist $inf([a], [b]) = [a] \cap [b]$ and $sup([a], [b]) = [a] \cup [b]$, i.e. if $(L/E_Q; \leq_Q)$ is a lattice.

An $infsup$-quasiordered set $(L; Q)$ is called complete, if for each $\emptyset \neq Y \in L/E_Q$ there exist $inf(Y) \in L/E_Q$ and $sup(Y) \in L/E_Q$, i.e. if $(L/E_Q; \leq_Q)$ is a complete lattice.

Let (L, Q) be an $infsup$-quasiordered set, $K : L/E_Q \to L$ is an arbitrary choice function and for any two elements $x, y \in L$ we have:$x \cap y = K(sup([x], [y])), x \cup y = K(inf([x], [y]))$, then the algebra $(L; \cap, \cup)$ is a q-lattice.

Let $(L; \cap, \cup)$ be a q-lattice, then the relation $aQb \leftrightarrow a \cap b = a \cap a$ is a quasiorder on the set L, the function $K : L/E_Q \to L$, which is defined in the following manner $K([a]) = a \cap a$ is a choice function and the pair (L, Q) is an $infsup$-quasiordered set, where $inf([a], [b])$ and $sup([a], [b])$ are defined by the following rules: $inf([a], [b]) = [a \cap b], sup([a], [b]) =$

$[a \cup b]$. Moreover, for the operations $\cap$ and $\cup$ we have: $x \cap y = K(inf([x],[y])), x \cup y = K(sup([x],[y]))$.

The function $\varphi : L \to L$ of a complete $infsup$-quasiordered set $(L;Q)$ is called monotone if it follows from xQy that $\varphi(x)Q\varphi(y)$.

The function $\varphi : L \to L$ of a complete $infsup$-quasiordered set $(L;Q)$ is called homomorphism, if φ is a homomorphism of the corresponding q-lattice $(L;\cap,\cup)$ into itself.

The point x of an $infsup$-quasiordered set (L,Q) is called a fixed point of the function $\varphi : L \to L$, if $\varphi(x) = x$.

The function $\varphi : L \to L$ of a complete $infsup$-quasiordered set $(L;Q)$ is called antimonotone, if it follows from xQy that $\varphi(y)Q\varphi(x)$.

The function $\varphi : L \to L$ of a complete $infsup$-quasiordered set $(L;Q)$ is called antihomomorphism, if the function φ is a homomorphism for the corresponding q-lattice $(L;\cap,\cup)$ into itself.

Note, that if $\varphi : L \to L$ is a homomorphism (an antihomomorphism) of an $infsup$-quasiordered set (L;Q) into itself, then the induced function $\tilde{\varphi} : L/E_Q \to L/E_Q$, which is defined in the following manner $\tilde{\varphi}([x]) = [\varphi(x)]$ is a homomorphism (an antihomomorphism).

The points x, y of an $infsup$-quasiordered set (L,Q) with the property xQy are called alternative fixed points of the function $\varphi : L \to L$, if $\varphi(x) = y$ and $\varphi(y) = x$.

The alternative fixed points x, y of the function $\varphi : L \to L$ of an $infsup$-quasiordered set (L,Q) into itself are called extreme, if for each alternative fixed points a, b of the function φ we have $xQaQbQy$.

Main result

Theorem 1) Each homomorphism $\varphi : L \to L$ of the complete $infsup$-quasiordered set $(L;Q)$ has alternative fixed points. Moreover, if $[\alpha] = sup\{[x] \in L/E_Q | [x] \leq_Q \tilde{\varphi}([x])\} \in L/E_Q$ is the greatest fixed point of the function $\tilde{\varphi}$, then for each fixed point a of the function φ we have $aQ(\alpha \cap \alpha)$. Similarly, if $[\beta] = inf\{[x] \in L/E_Q | [x] \leq_Q \tilde{\varphi}([x])\} \in L/E_Q$ is the lower fixed point of the function $\tilde{\varphi}$, then for any fixed point a of the function φ we have $(\beta \cap \beta)Qa$. Moreover, the $infsup$-quasiordered set $Fix(\varphi) = \{x \in L | \varphi(x) = x\}$ is a complete $infsup$-quasiordered set.

2) Each antihomomorphism $\varphi : L \to L$ of the complete $infsup$-quasiordered set $(L;Q)$ has extreme alternative fixed points.

An application of this theorem in semantic of logic programming is considered.

References

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