Mathematical Problems of Computer Science 54, 7–17, 2020.

UDC 519.1

A Note on Hamiltonian Bypasses in Digraphs with Large Degrees

Samvel Kh. Darbinyan

Institute for Informatics and Automation Problems of NAS RA
e-mail: samdarbin@iiap.am

Abstract

Let D be a 2-strongly connected directed graph of order $p \geq 3$. Suppose that $d(x) \geq p$ for every vertex $x \in V(D) \setminus \{x_0\}$, where x_0 is a vertex of D. In this paper, we show that if D is Hamiltonian or $d(x_0) > 2(p-1)/5$, then D contains a Hamiltonian path, in which the initial vertex dominates the terminal vertex.

Keywords: Digraph, cycle, Hamiltonian cycle, Hamiltonian bypass.

1. Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. Every cycle and path are assumed to be simple and directed. We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1]. A cycle (path) in a digraph D passing through all the vertices of D is called *Hamiltonian*. A digraph containing a Hamiltonian cycle is called a *Hamiltonian digraph*. A Hamiltonian path in a digraph D in which the initial vertex dominates the terminal vertex is called a Hamiltonian bypass. There are numerous sufficient conditions for the existence of a Hamiltonian cycle in digraphs (see, e.g., [1, 2, 3]). It is natural to consider an analogous problem for the existence of a Hamiltonian bypass.

It was proved in [4] - [9] that a number of sufficient conditions for a digraph to be Hamiltonian is also sufficient for a digraph to contain a Hamiltonian bypass (with some exceptions which are characterized). In particular, Theorems 1.4 and 1.5 were proved in [5] and [6], respectively. To formulate these theorems, we need the following definitions.

Definition 1: *Let D_0 denote any digraph of order $p \geq 3$, p is odd, such that $V(D_0) = A \cup B$, where $A \cap B = \emptyset$, A is an independent set with $(p+1)/2$ vertices, B is a set of $(p-1)/2$ vertices inducing an arbitrary subdigraph, and D_0 contains all the possible arcs between A and B.*

Definition 2: *For any $k \in [1, p-2]$ let $D_{p-k,k}$ denote a digraph of order $p \geq 3$, obtained from K^*_{p-k} and K^*_{k+1} by identifying a vertex of the first with a vertex of the second.*

Definition 3: *By T_5 we denote a tournament of order 5 with vertex set $\{x_1, x_2, x_3, x_4, y\}$ and arc set $\{x_ix_{i+1} \,|\, i \in [1,3]\} \cup \{x_4x_1, x_1y, x_3y, yx_2, yx_4, x_1x_3, x_2x_4\}$.*

Theorem 1: (Benhocine [5]). *Let D be a 2-strong digraph of order p with minimum degree at least $p-1$. Then D contains a Hamiltonian bypass, unless D is isomorphic to a digraph of type D_0.*

Theorem 2: (Darbinyan [6]). *Let D be a strong digraph of order $p \geq 3$. Suppose that $d(x)+d(y) \geq 2p-2$ for every pair of non-adjacent vertices x, y of $V(D)$. Then D contains a Hamiltonian bypass, unless D is isomorphic to a digraph of the set $D_0 \cup \{D_{p-k,k}, T_5, C_3\}$.*

The author [10] proved the following results.

Theorem 3: (Darbinyan [10]). *For every integer $p \geq 8$ there is a 2-strong non-Hamiltonian digraph of order p, which has $p-1$ vertices of degrees at least p.*

Theorem 4: (Darbinyan [9]). *Let D be a 2-strong digraph of order $p \geq 3$ with the minimum degree at least $p-4$. If $p-1$ vertices of D have degrees at least p, then D is Hamiltonian.*

Theorem 5: (Darbinyan [10]). *Let D be a strong digraph of order $p \geq 3$. Suppose that $d(x)+d(y) \geq 2p-1$ for every pair of non-adjacent vertices $x, y \in V(D) \setminus \{z_0\}$, where z_0 is some vertex in $V(D)$. Then D contains a cycle of length at least $p-1$.*

The following corollary follows from Theorem 5.

Corollary 1: *Let D be a strong digraph of order $p \geq 3$. If $p-1$ vertices of $V(D)$ have degrees at least p, then D is Hamiltonian or contains a cycle of length $p-1$ (in fact, D has a cycle that contains all the vertices with degrees at least p).*

Remark 1: For the proof of Theorem 3, it suffices to consider a digraph $H(n)$ of order $n \geq 8$, which is defined as follows:

$$V(H(n)) := \{x_0, x_1, x_2, \ldots, x_{n-4}, y_1, y_2, y_3\} \quad \text{and}$$

$$A(H(n)) := \{y_iy_j \,|\, i \neq j\} \cup \{x_ix_{i+1} \,|\, 0 \leq i \leq n-4\} \cup \{y_ix_j \,|\, 1 \leq i \leq 3, 1 \leq j \leq n-6\}$$

$$\cup \{x_ix_j \,|\, 1 \leq j < i \leq n-4\} \cup \{x_{n-4}y_i, x_{n-6} \,|\, 1 \leq i \leq 3\} \cup \{x_ix_{n-5} \,|\, 1 \leq i \leq n-7\}$$

$$\cup \{x_0x_{n-5}, x_{n-5}x_0, x_{n-4}x_0, x_{n-6}x_{n-4}\}.$$

Note that Theorem 3 disproves a conjecture of Thomassen ([2]. *Every 3-strong digraph of order p with minimum degree at least $p+1$ is strongly Hamitonian-connected*).

In this paper, we prove the following theorem.

Theorem 6: *Let D be a 2-strong digraph of order $p \geq 3$. Suppose that $d(x) \geq p$ for every vertex $x \in V(D) \setminus \{x_0\}$, where x_0 is a vertex of D. If D is Hamiltonian or $d(x_0) > 2(p-1)/5$, then D contains a Hamiltonian bypass.*

2. Terminology and Notation

In this paper, we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The *order* of D is the number of its vertices. Let x, y be distinct vertices in D. The arc of a digraph D directed from x to y is denoted by xy (we say that x dominates y). For disjoint subsets A and B of $V(D)$ we define $A(A \to B)$ as the set $\{xy \in E(D) \,|\, x \in A, y \in B\}$, $A(A, B) = A(A \to B) \cup A(B \to A)$. The notation $A \to B$ denotes that every vertex of A dominates every vertex of B. $A \mapsto B$ means that $A \to B$ and there is no arc from a vertex of B to a vertex of A. If $x \in V(D)$ and $A = \{x\}$, we write x instead of $\{x\}$. The *out-neighborhood* of a vertex x is the set $N^+(x) = \{y \in V(D) \,|\, xy \in A(D)\}$ and $N^-(x) = \{y \in V(D) \,|\, yx \in A(D)\}$ is the *in-neighborhood* of x. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A \,|\, xy \in A(D)\}$ and $N^-(x, A) = \{y \in A \,|\, yx \in A(D)\}$. The *out-degree* of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The *degree* of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$).

The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D[A]$. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $1 \leq i \leq m-1$ (respectively, $x_i x_{i+1}$, $1 \leq i \leq m-1$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a *path from* x_1 *to* x_m or is an (x_1, x_m)*-path*. The *length* of a cycle or a path is the number of its arcs. A cycle of length k, $k \geq 2$, is denoted by C_k. For a cycle $C_k := x_1 x_2 \cdots x_k x_1$, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod k$. If P is a path containing a subpath from x to y, we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. For a digraph D of order n, by $D(n, 2) = [x_1 x_n; x_1 x_2 x_3 \ldots x_n]$ we denote a Hamiltonian path in which the initial vertex x_1 dominates the terminal vertex x_n.

A digraph D is *strongly connected* (or, just, *strong*) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. A digraph D is k*-strongly connected* (or, k*-strong*), if $|V(D)| \geq k+1$ and $D[V(D) \setminus A]$ is strong for any set A of at most $k-1$ vertices. Two distinct vertices x and y of a digraph D are *adjacent* if $xy \in A(D)$ or $yx \in A(D)$ (or both). By K_n^* is denoted the complete digraph of order n.

3. Preliminaries

The following well-known simple Lemmas 1-3 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proof of our result.

Lemma 1: (Häggkvist and Thomassen [12]). *Let D be a digraph of order $p \geq 3$ containing a cycle C_m, $2 \leq m \leq p-1$. Let x be a vertex not contained in this cycle. If $d(x, V(C_m)) \geq m+1$, then for every k, $2 \leq k \leq m+1$, D contains a cycle of length k including x.*

The following lemma is a modification of a lemma by Bondy and Thomassen [13].

Lemma 2: *Let D be a digraph of order $p \geq 3$ containing a path $P := x_1 x_2 \ldots x_m$, $2 \leq m \leq p-1$ and x be a vertex not contained in this path. If one of the following conditions holds:*

(i) $d(x, V(P)) \geq m+2$;
(ii) $d(x, V(P)) \geq m+1$ *and* $xx_1 \notin A(D)$ *or* $x_m x \notin A(D)$;
(iii) $d(x, V(P)) \geq m$, $xx_1 \notin A(D)$ *and* $x_m x \notin A(D)$;
then there is an i, $1 \leq i \leq m-1$, *such that* $x_i x, xx_{i+1} \in A(D)$ *i.e.,* $x_1 x_2 \dots x_i x x_{i+1} \dots x_m$ *is a path of length* m D *(we say that* x *can be inserted into* P*).*

The following lemma is a simple extension of a lemma by Bang-Jensen, Gutin and Li [14].

Lemma 3: *Let* $P = u_1 u_2 \dots u_s$ *be a path in a digraph* D *(possibly,* $s = 1$*) and let* $Q = v_1 v_2 \dots v_t$ *be a path (or* $Q = v_1 v_2 \dots v_t v_1$ *be a cycle) in* $D[V(D) \setminus V(Q)]$, $t \geq 2$. *Suppose that for each* u_i, $1 \leq i \leq s$, *there is an arc* $v_j v_{j+1}$ *on* Q *such that* $v_j u_i, u_i v_{j+1} \in A(D)$. *Then there is a* (v_1, v_t)*-path (or a cycle) of length* $t+k-1$ *(respectively,* $t+k$*),* $1 \leq k \leq s$, *with vertex set* $\{v_1, v_2, \dots, v_t\} \cup \{u_1, u_2, \dots, u_k\}$.

4. Proofs of the Main Results

Theorem 6: *Let* D *be a 2-strong digraph of order* $p \geq 3$. *Suppose that* $d(x) \geq p$ *for every vertex* $x \in V(D) \setminus \{x_0\}$, *where* x_0 *is a vertex of* D. *If* D *is Hamiltonian or* $d(x_0) > 2(p-1)/5$, *then* D *contains a Hamiltonian bypass.*

Proof: Suppose, on the contrary, that is D contains no Hamiltonian bypass. We first will prove the following claim (note that in the proofs of Claim 1 and Case 1, we do not use the fact that D is 2-strong).

Claim 1: *D has no cycle of length l through x_0, where $l = p-1$ or $l = p-2$.*
Proof: Suppose that the claim is not true. Assume that $C_{p-1} = x_1 x_2 \dots x_{p-1} x_1$, $x_0 \in V(C_{p-1})$ and $y \notin V(C_{p-1})$. Since D contains no Hamiltonian bypass, for every i, $1 \leq i \leq p-1$, we have $d^+(y, \{x_i, x_{i+1}\}) \leq 1$ and $d^-(y, \{x_i, x_{i+1}\}) \leq 1$. Therefore,

$$2d(y) = \sum_{i=1}^{p-1} (d^+(y, \{x_i, x_{i+1}\}) + d^-(y, \{x_i, x_{i+1}\})) \leq 2(p-1),$$

which contradicts that $d(y) \geq p$. Thus, D contains no cycle of length $p-1$ through x_0.

Now assume that D contains a cycle of length $p-2$ through x_0. Let $C_{p-2} = x_1 x_2 \dots x_{p-2} x_1$, $x_0 \in V(C_{p-2})$ and $x, y \notin V(C_{p-2})$. Since D contains no cycle of length $p-1$ through x_0, from Lemma 1 it follows that $xy, yx \in A(D)$, $d(x, V(C_{p-2})) = d(y, V(C_{p-2})) = p-2$ and there is a vertex x_i such that the vertices x, x_i are not adjacent and the arcs $x_{i-1}x, xx_{i+1}$ are in D. If $yx_i \in A(D)$, then $D(p,2) = [yx_i; yxC[x_{i+1}, x_i]]$, if $x_i y \in A(D)$, then $D(p,2) = [x_i y; C[x_i, x_{i-1}]xy]$, a contradiction. We may therefore assume that x_i and y also are not adjacent. Using this, Lemmas 1, 2 and the fact that D contains no cycle of length $p-1$ througt x_0, we obtain that $x_i = x_0$, $x_{i-1}y$, $yx_{i+1} \in A(D)$ and the vertex x (y) is adjacent to every vertex in $V(D) \setminus \{x_0\}$. Hence, we have that if $xx_{i+2} \in A(D)$, then $D(p,2) = [yx_{i+1}; yxC[x_{i+2}, x_{i+1}]]$, a contradiction. If $xx_{i+2} \notin A(D)$, then $x_{i-2}x \in A(D)$ and $D(p,2) = [x_{i-1}y; C[x_{i-1}, x_{i-2}]xy]$, a contradiction. Claim 1 is proved. □

Now, we divide the proof into two cases to consider.

Case 1. D is Hamiltonian.

Let $C_p = x_1x_2\dots x_px_1$ be a Hamiltonian cycle in D. Since D contains no Hamiltonian bypass, we have that $x_{i+1}x_i \notin A(D)$ for every i, $1 \le i \le p$. Using this, it is not difficult to check that if $p \le 6$, then D contains a Hamiltonian bypass. We may therefore assume that $p \ge 7$.

Claim 2: *If $x_0 \neq x_{i+1}$, then the vertices x_i and x_{i+2} are not adjasent, where $1 \le i \le p$.*

Proof: Suppose, on the contrary, that is for some i, $1 \le i \le p$, $x_0 \neq x_{i+1}$ and the vertices x_i, x_{i+2} are adjasent. Without loss of generality we may assume that $i = 1$. Since $x_0 \neq x_2$, we have $d(x_2) \ge p$ and, by Claim 1, $x_1x_3 \notin A(D)$. Hence, $x_3x_1 \in A(D)$. It is clear that $x_0 \neq x_1$ or $x_0 \neq x_3$. This together with Claim 1 implies that $x_px_2 \notin A(D)$ or $x_2x_4 \notin A(D)$. Since there is no (x_3, x_1)-Hamiltonian path, using Lemma 2(ii) , we obtain that

$$d(x_2, V(D) \setminus \{x_2\}) = d(x_2, \{x_1, x_3\}) + d(x_2, V(D) \setminus \{x_1, x_2, x_3\}) \le 2 + p - 3 = p - 1,$$

which contradicts that $d(x_2) \ge p$. □

It is not difficult to show that there are two distinct vertices x_i and x_{i+k} such that $x_{i+k}x_i \in A(D)$ and $x_0 \notin \{x_{i+1}, x_{i+2}, \dots, x_{i+k-1}\}$. We may assume that k is chosen so that k is the smallest possible. Without loss of generality we may assume that $i = 1$. Then $d^-(x_1, \{x_2, x_3, \dots, x_k\}) = 0$. From Claim 2 it follows that $3 \le k \le p - 2$.

Assume first $k = 3$, i.e., $x_4x_1 \in A(D)$. By Claim 2, the vertices x_2 and x_4 (x_1 and x_3) are not adjacent since $x_0 \notin \{x_2, x_3\}$. Now from $x_{i+1}x_i \notin A(D)$, $d(x_2) \ge p$ and $d(x_3) \ge p$ it follows that

$$d(x_2, \{x_5, x_6, \dots, x_p\}) \ge p - 2 \quad \text{and} \quad d(x_3, \{x_5, x_6, \dots, x_p\}) \ge p - 2.$$

Hence, by Lemma 2, the vertex x_2 (x_3) can be inserted into $x_5x_6\dots x_p$. Then, by Lemma 3, there is an (x_4, x_1)-Hamiltonian path, which is a contradiction as $x_4x_1 \in A(D)$.

Assume next that $k \ge 4$. By Claim 2, the vertices x_i and x_{i+2}, where $1 \le i \le k - 1$ are not adjacent. From the minimality of k it follows that if $1 \le i < j \le k+1$, then $x_jx_i \in A(D)$ if and only if $j = k + 1$ and $i = 1$. From the minimality of $k \ge 4$ and Claim 1 it follows that for each $x_i \in \{x_1, x_2, \dots, x_{k-2}\}$,

$$d(x_i, \{x_{i+2}, x_{i+3}\}) = d(x_{k-1}, \{x_{k+1}\}) = 0. \tag{1}$$

Also we need to show the following claim.

Claim 3: *Suppose that $1 \le i < j - 1 \le k$. Then $x_ix_j \in A(D)$ if and only if $i = 1$ and $j = k + 1$.*

Proof: For a proof by contradiction, suppose that $x_mx_n \in A(D)$, where $1 \le m < n - 1 \le k$ and $m \neq 1$ or $n \neq k+1$. Without loss of generality, we may assume that $n-m$ is the minimum possible. From (1) it follows that $n - m \ge 4$, i.e., $|\{x_{m+1}, \dots, x_{n-1}\}| = n - m - 1 \ge 3$. Note that $R := x_mx_nx_{n+1}\dots x_px_1x_2\dots x_m$ is a cycle of length $p - n + m + 1 \le p - 3$ through x_0. By the minimality of k and $n - m$, for every $y \in \{x_{m+1}, \dots, x_{n-1}\}$ we have

$$d(y, \{x_{m+1}, \dots, x_{n-1}\}) \le 2 \quad \text{and} \quad d(y, V(R)) \ge p - 2.$$

Therefore, by Lemma 1, every vertex $y \in \{x_{m+1}, \dots, x_{n-1}\}$ can be inserted into R. Now using Lemma 3, we obtain a cycle of length $p - 1$ through x_0, which contradicts Claim 1. □

From Claim 3 and the minimality of $k \ge 4$ it follows that

$$d(x_2, \{x_2, x_3, \dots, x_k\}) = d(x_k, \{x_2, x_3, \dots, x_k\}) = 1$$

and for every i, $3 \le i \le k-1$, $d(x_i, \{x_2, x_3, \dots, x_k\}) = 2$. Therefore, $d(x_i, V(Q)) \ge p-2$, where $2 \le i \le k$ and $Q := x_{k+1}x_{k+2}\dots x_p x_1$. Note that $|V(Q)| = p-k+1$. If $k \ge 5$, then $|V(Q)| \le p-4$, and, by Lemma 2(i), every vertex x_i, $2 \le i \le k$, can be inserted into Q. If $k = 4$, then $|V(Q)| = p-3$, $d(x_2, V(Q)) \ge p-1$, $d(x_4, V(Q)) \ge p-1$ and $d(x_3, V(Q)) \ge p-2$. Since $d(x_3, \{x_1, x_5\}) = 0$, again using Lemma 2, we obtain that each vertex $x_i \in \{x_2, x_3, x_4\}$ can be inserted into Q. Therefore, by Lemma 3, there is an (x_{k+1}, x_1)-Hamiltonian path, which contradicts our initial supposition since $x_k x_1 \in A(D)$. The discussion of Case 1 is completed.

Case 2. D is not Hamiltonian.

(*) Observe that by Claime 1, in this case every cycle through x_0 in D has length at most $p-3$.

Then, by Corollary 1, D contains a cycle of length $p-1$. Let $C_{p-1} = x_1x_2\dots x_{p-1}x_1$ be a cycle of length $p-1$ in D. By Claim 1, $x_0 \in V(C_{p-1})$. For this case, we first give the following claim and lemma.

Claim 4: *Let $P := x_1x_2\dots x_{p-1}$ be an (x_1, x_{p-1})-path of length $p-2$ through x_0 in D. Then $x_1x_{p-1} \notin A(D)$.*

Proof: For a proof by contradiction, suppose that $x_1x_{p-1} \in A(D)$. Let $x \notin V(P)$. Then $d(x) \ge p$ since $x \ne x_0$. Since D contains no Hamiltonian bypass, it follows that x cannot be inserted into P. Now using Lemma 2(i) and $d(x) \ge p$, we obtain that $x_{p-1}x$ and $xx_1 \in A(D)$. Therefore, $x_1x_2\dots x_{p-1}xx_1$ is a Hamiltonian cycle in D, which contradicts the hypothesis of this case. □

Lemma 4: *D contains no cycle of length $p-3$ through x_0.*

Proof: Suppose that the lemma is not true. Let $C := x_1x_2\dots$
$x_{p-4}x_0x_1$ be a cycle of length $p-3$ through x_0 in D and let $B := V(D) \setminus V(C)$. By Claim 1, D contains no cycle of length $p-1$ and $p-2$ through x_0. This together with Lemma 1 implies that for every $y \in B$,

$$p \le d(y) = d(y, V(C)) + d(y, B) \le p-3+d(y, B).$$

Therefore, $d(y, B) \ge 3$. This implies that $D[B]$ is Hamiltonian since $|B| = 3$, in particular, $D[B]$ is strong.

We now consider the folowing two cases.

Case (a). There exists a vertex $y \in B$, which is adjacent to every vertex x_i for all i, $1 \le i \le p-4$.

Let $yuzy$ be a Hamiltonian cycle in $D[B]$.

If y and x_0 are adjacent then using the observation (*), it is not difficult to show that either $d^-(y, V(C)) = 0$ or $d^+(y, V(C)) = 0$. Without loss of generality, we assume that $d^+(y, V(C)) = 0$. Then $V(C) \mapsto y$. This together with Claim 1 implies that $A(B \to V(C)) = \emptyset$, which contradicts that D is 2-strong. We may therefore assume that y and x_0 are not adjacent. If $x_1y \in A(D)$, then $\{x_1, x_2, \dots, x_{p-4}\} \to y$. Therefore, $A(B \to V(C) \setminus \{x_1\}) = \emptyset$. This means that $D[V(D) \setminus \{x_1\}]$ is not strong, i.e., D is not 2-strong, a contradiction. Now assume that $x_1y \notin A(D)$. Then $yx_1 \in A(D)$ since y and x_1 are adjacent. Similarly, $x_{p-4}y \in A(D)$. Then by the above observation (*), $d(x_0, B) = 0$. Let $x_ky \in A(D)$ with $2 \le k \le p-4$ and k be the minimum possible. It is not dificult to show that

$$\{x_k, x_{k+1}, \dots, x_{p-4}\} \to y \to \{x_1, x_2, \dots, x_{k-1}\}.$$

Assume first that $k \le p-5$. Then by Claim 1, $d^-(x_{p-4}, B) = 0$. If $x_{p-4}z \in A(D)$, then $D(p, 2) = [x_{p-4}z; x_{p-4}x_0x_1\dots x_{p-5}yuz]$, a contradiction. Therefore, $x_{p-4}z \notin A(D)$. Thus,

$d(z, \{x_0, x_{p-4}\}) = 0$ and $d(z, \{x_1, x_2, \ldots, x_{p-5}\}) \geq p-4$. Again using Lemma 2(i) and (*), we obtain that $x_{p-5}z \in A(D)$. Therefore, $x_{p-4}x_0x_1 \ldots x_{p-5}zy$ is a path of length $p-2$ through x_0 and $x_{p-4}y \in A(D)$, which contradicts Claim 4.

Assume next that $k = p-4$. Then $y \to \{x_1, x_2, \ldots, x_{p-5}\}$. Hence, if $p-5 \geq 2$, then for the converse digraph of D we have the considered former case. For $4 \leq p \leq 6$, this completes the discussion of Case (a).

Case (b). For every $y \in B$ there exists a vertex x_k with $1 \leq k \leq p-4$ such that y and x_k are not adjacent.

To complete the proof of Lemma 4 in this case, we first prove the following claims.

Claim 5: *$x_{k-1}y \notin A(D)$ or $yx_{k+1} \notin A(D)$.*

Proof: Suppose, on the contrary, that $x_{k-1}y \in A(D)$ and $yx_{k+1} \in A(D)$. Note that $d(x_k) \geq p$ since $x_k \neq x_0$. Using observation (*), we obtain that $d(x_k, B) = 0$. Now consider the cycle $R := x_0x_1 \ldots x_{k-1}yx_{k+1} \ldots x_{p-4}x_0$ of length $p-3$ through x_0. By Claim 1, x_k cannot be inserted into R (for otherwise we obtain a cycle of length $p-2$ througth x_0). Therefore by Lemma 1, $p \leq d(x_k) = d(x_k, B) + d(x_k, V(R)) \leq p-3$, a contradiction. □

Claim 6: (i) *If $x_{k-1}y \in A(D)$, then $x_{k+1}y \notin A(D)$.* (ii) *If $yx_{k-1} \in A(D)$, then $yx_{k+1} \notin A(D)$.*

Proof: (i) Suppose that the claim is not true. Then $\{x_{k-1}, x_{k+1}\} \to y$. By Claim 5, $yx_{k+1} \notin A(D)$. Since y cannot be inserted into the path $C[x_{k+1}, x_{k-1}]$ and $yx_{k+1} \notin A(D)$, from Lemma 2(ii) it follows that $d(y, V(C)) = p-4$.

Assume first that there is a vertex $x_s \neq x_k$ such that y and x_s also are not adjacent. Let s be chosen so that $|V(C[x_k, x_s])|$ is the minimum possible. Note that $x_s \notin \{x_{k-1}, x_{k+1}\}$.

Write $P_1 := C[x_{k+1}, x_{s-1}]$ and $P_2 := C[x_{s+1}, x_{k-1}]$. Then

$$p-4 = d(y, V(C)) = d(y, V(P_1) + d(y, V(P_2) \leq |V(P_1)| + |V(P_2)| + 1 = p-4.$$

This implies that $d(y, V(P_1)) = |V(P_1)|$ and $d(y, V(P_2)) = |V(P_2)| + 1$. Now using Lemma 2, we obtain $x_{s-1}y \in A(D)$ and $yx_{s+1} \in A(D)$. By Claim 5, $x_s = x_0$ and $d^-(x_{k+1}, B) = d(x_s, B) = 0$. Rcall that $zyuz$ is a Hamiltonian cycle in $D[B]$. If $x_kz \in A(D)$, then $D(p, 2) = [x_kz; C[x_k, x_{k-1}]yuz]$, a contradiction. Therefore, z and x_k are not adjacent. Now using Lemma 2, $d(z) \geq p$, $d(z, \{x_k, x_s\}) = 0$ and the fact that $d^+(z, \{x_{k+1}, x_{k+2}\}) = 0$, we obtain $x_{k+1}z \in A(D)$. Therefore, $C[x_{k+1}, x_{k-1}]yuz$ is a path of length $p-2$ thruogh x_0 and $x_{k+1}z \in A(D)$, which contradicts Claim 4.

Assume next that y is adjacent to every vertex of $V(C) \setminus \{x_k\}$. Then by Claim 1, $V(C) \setminus \{x_k\} \mapsto y$ since $x_{k+1}y \in A(D)$ and $yx_{k+1} \notin A(D)$. Again using observation (*), we obtain that $A(B \to V(C)) = \emptyset$, which contradicts that D is 2-strong. This completes the proof of Claim 6(i).

For the proof of Claim 6(ii), it suffices to consider the converse digraph of D. Claim 6 is proved. □

Claim 7: *If $yx_{k-1} \in A(D)$, then $x_{k+1}y \notin A(D)$.*

Proof: For a proof by contradiction, suppose that $yx_{k-1} \in A(D)$ and $x_{k+1}y \in A(D)$. Claim 6 implies that $x_{k-1}y \notin A(D)$ and $yx_{k+1} \notin A(D)$. Since y cannot be inserted into $C[x_{k+1}, x_{k-1}]$, using Lemma 2(iii), we obtain $d(y, V(C[x_{k+1}, x_{k-1}]) \leq p-5$, which contradicts that $d(y, V(C)) \geq p-4$. Claim 7 is proved. □

Claim 8: (i) *The vertices y and x_{k+1} are adjacent*; (ii) *The vertices y and x_{k-1} are adjacent.*

Proof: (i) Suppose that the claim is not true, i.e., $d(y, \{x_k, x_{k+1}\}) = 0$. Write $Q :=$

$C[x_{k+2}, x_{k-1}]$. Then $d(y, V(Q)) = p-4$. Therefore by Lemma 2, $yx_{k+2} \in A(D)$ and $x_{k-1}y \in A(D)$ since y cannot be inserted into Q.

Assume first that $x_{k+1} \neq x_0$. We know that $d(x_k) \geq p$ and $d(x_{k+1}) \geq p$. Using obveration (*), it is not difficult to show that $d(x_k, B) = d(x_{k+1}, B) = 0$. Therefore, $d(x_k, V(Q)) \geq p-2$ and $d(x_{k+1}, V(Q)) \geq p-2$. These together with Lemmas 2 and 3 imply that the vertices x_k and x_{k+1} both can be inserted into Q. As a consequence, we obtain a cycle of length $p-2$ through x_0, which contradicts Claim 1.

Assume next that $x_{k+1} = x_0$. Then $d(x_0, B) = d^-(x_k, B) = 0$. If $x_k z \in A(D)$, then $D(p,2) = [x_k z; C[x_k, x_{k-1}]yuz]$, a contradiction. If $x_k u \in A(D)$, then $C[x_k, x_{k-1}]yz$ is a path of length $p-2$ through x_0 and $x_k u \in A(D)$, which contradicts Claim 4. We may therefore assume that $d(z, \{x_k, x_{k+1}\}) = d(u, \{x_k, x_{k+1}\}) = 0$. Therefore, $d(z, V(Q)) = p-4$, zx_{k+2} and $x_{k-1}z \in A(D)$. Now using Claims 1 and 5, we obtain that there is a vertex x_s such that $\{y, z\} \to V(C[x_{k+2}, x_s])$ and $V(C[x_s, x_{k-1}]) \to \{y, z\}$. Whitout loss of generality, assume that $|V(C[x_{k+2}, x_s])| \geq 2$ (for otherwise we consider the converse digraph of D). Then $D(p,2) = [yx_{k+2}; yuzC[x_{k+3}, x_{k+2}]]$, a contradiction. This contradiction completes the proof of Claime 8(i). By the same arguments one can prove Claim 8(ii). Claim 8 is proved. □

Now we return to the proof of the lemma. From Claim 8 it follows that y is adjacent to x_{k-1} and x_{k+1}. Therefore, only the following cases are possible: (i) $x_{k-1}y$ and $yx_{k+1}y \in A(D)$, (ii) $\{x_{k-1}, x_{k+1}\} \to y$, (iii) $y \to \{x_{k-1}, x_{k+1}\}$, (iv) $x_{k+1}y$ and $yx_{k-1} \in A(D)$. On the other hand, Claims 5, 6 and 7 imply that none of these cases holds. This contradiction completes the discussion of Case (b). Lemma 4 is proved. □

Now we are ready to complete the proof of the theorem in Case 2. Since D is not Hamiltonian, by Corollary 1, D contains a cycle of length $p-1$. Let $R := x_1x_2\ldots x_{p-1}x_1$ be a cycle of length $p-1$ in D. Then by Claim 1 and Lemma 4, we know $x_0 \notin V(R)$ and for every i, j, $1 \leq i, j \leq p-1$ the following hold:

$$d^-(x_0, \{x_i\}) + d^+(x_0, \{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}) \leq 1,$$

$$d^+(x_0, \{x_j\}) + d^-(x_0, \{x_{j-1}, x_{j-2}, x_{j-3}, x_{j-4}\}) \leq 1.$$

Therefore, $d^-(x_0) + 4d^+(x_0) \leq p-1$ and $4d^-(x_0) + d^+(x_0) \leq p-1$. These mean that $5d(x_0) \leq 2p-2$, i.e., $d(x_0) \leq 2(p-1)/5$, which contradicts that $d(x_0) > 2(p-1)/5$. The theorem is proved. □

Corollary 2: (Benhocine [5]). *Every strong digraph D of order $p \geq 3$ and with minimum degree at least p contains $D(p,2)$.*

Proof: By the famous theorem of Ghoula-Houri, D is Hamiltonian. Therefore, from the proof of Theorem 6 in Case (a), it follows that D contains a Hamiltonian bypass. □

Perhaps the following proposition will be useful for Conjecture 1 (see, in section Conclusion).

Proposition 1: *Let D be a non-Hamiltonian 2-strong digraph of order $p \geq 3$. Suppose that $d(x) \geq p$ for every vertex $x \in V(D) \setminus \{x_0\}$, where x_0 is a vertex of D. If $P = x_1x_2\ldots x_{p-2}$ is an (x_1, x_{p-2})-path of length $p-3$ through x_0 in D, then $x_1x_{p-2} \notin A(D)$.*

Proof: For a proof by contradiction, suppose that $x_1x_{p-2} \in A(D)$. Write $V(D) \setminus V(P) = \{y_1, y_2\}$. We know that $d(y_1) \geq p$ and $d(y_2) \geq p$ since $x_0 \in V(P)$. From Claim 4 it follows that y_i cannot be inserted into P. On the other hand, since D contains no cycle of length $p-1$ through x_0, we have that $y_ix_1 \notin A(D)$ or $x_{p-2}y_i \notin A(D)$. Now using Lemma 2(ii), we obtain $d(y_i, V(P)) = p-2$ and y_1y_2, $y_2y_1 \in A(D)$. Without loss of generality, assume that $y_1x_1 \notin A(D)$. Then by Lemma 2(ii), $x_{p-2}y_1 \in A(D)$. Since D is not Hamiltonian and contains no cycle of length $p-1$ through x_0, it follows that $d^-(x_1, \{y_1, y_2\}) = 0$. Then $x_{p-2}y_2 \in A(D)$. If $x_1y_1 \in A(D)$ (or $x_1y_2 \in A(D)$), then it is not difficult to show that $D(p, 2) = [x_1y_1; x_1x_2 \ldots x_{p-2}y_2y_1]$ (or $D(p, 2) = [x_1y_2; x_1x_2 \ldots x_{p-2}y_1y_2]$) is in D, a contradiction. Therefore, $d^+(x_1, \{y_1, y_2\}) = 0$. Thus, $d(x_1, \{y_1, y_2\}) = 0$. This together with Lemma 2 and $d(y_i, V(P)) = p-2$ implies that $\{y_1, y_2\} \to x_2$. Then by Claim 1, $x_1 = x_0$ since $x_2x_3 \ldots x_{p-2}y_1y_2x_2$ is a cycle of length $p-1$.

Write $Q := x_2x_3 \ldots x_{p-2}$. Then $|V(Q)| = p-3$, $d(y_1, V(Q)) \geq p-2$ and $d(y_2, V(Q)) \geq p-2$. Since $x_0 \to \{x_2, x_{p-2}\}$, by Claim 4 we have that neither y_1 nor y_2 can be inserted into Q. Then by Lemma 2(iii), we obtain that $d(y_1, V(Q)) = d(y_2, V(Q)) = p-2$ and the arcs y_2y_1, $x_{p-2}y_2$, y_1x_2 are in $A(D)$.

We claim that the vertex y_1 (y_2) is adjacent to each vertex of $V(Q)$.

Assume that this is not the case. Let $d(y_1, \{x_i\}) = 0$, where $3 \leq i \leq p-3$. From Lemma 2(iii), $d(y_1, V(Q)) = p-2$ and the fact that the vertex y_1 cannot be inserted into Q it follows that $x_{i-1}y_1, y_1x_{i+1} \in A(D)$. Since y_2y_1, $y_1y_2 \in A(D)$, it is easy to see that $d(y_2, \{x_i\}) = 0$ and $x_{i-1}y_2, y_2x_{i+1} \in A(D)$. They imply that the vertex x_i can be inserted neither into $S := x_2 \ldots x_{i-1}$ nor into $T := x_{i+1} \ldots x_{p-2}$. Then it is easy to see that $d(x_i, \{x_0\}) = 2$ and

$$p-2 \leq d(x_i, V(S)) + d(x_i, V(T)) \leq |V(S)| + |V(T)| + 2 = p-2.$$

Therefore, $d(x_i, V(S)) = |V(S)| + 1$ and $d(x_i, V(T)) = |V(T)| + 1$. Again using Lemma 2, we obtain $x_{p-2}x_i$ and $x_ix_2 \in A(D)$. Hence, $x_0x_ix_2 \ldots x_{i-1}y_1y_2x_{i+1} \ldots x_{p-2}$ is an (x_0, x_{p-2})-Hamiltonian path, a contradiction as $x_0x_{p-2} \in A(D)$. This proves that y_1 (y_2) is adjacent to every vertex in $V(Q)$. Therefore, there is an integer l, $2 \leq l \leq p-2$, such that

$$\{x_l, x_{l+1}, \ldots, x_{p-2}\} \to \{y_1, y_2\} \to \{x_2, x_3, \ldots, x_l\}. \tag{2}$$

If $3 \leq l \leq p-3$, then by (2), $x_0x_{p-2}y_1x_3 \ldots x_{p-3}y_2x_2$ is an (x_0, x_2)-Hamiltonian path, which is a contradiction as $x_0x_2 \in A(D)$. If $l = 2$ or $l = p-2$, then $A(\{y_1, y_2\} \to V(D) \setminus \{x_2\}) = \emptyset$ or $A(V(D) \setminus \{x_{p-2}\} \to \{y_1, y_2\}) = \emptyset$ when $l = 2$ or $l = p-2$, respectively. This means that D is not 2-strong, a contradiction. Proposition 1 is proved. □

5. Conclusion

In the current article, we examined the existence of a Hamiltonian bypass in 2-strong digraphs of order p, in which $p-1$ vertices have degrees at least p. We proved that if such digraphs are Hamiltohian or have the minimal degree more than $2(p-1)/5$, then such digraphs contain a Hamiltonian bypass.

If we consider the digraph $H(n)$ (by Remark 1, $H(n)$ is 2-strong, $d(x_0) = 4$ and is not Hamiltonian), then we see that $D(n, 2) = [y_1x_1; y_1y_2y_3x_2x_3 \ldots x_{n-4}x_0x_1]$ is a Hamiltonian bypass. By the above arguments, we believe that the following conjecture is true.

Conjecture 1: *Let D be a 2-strong digraph of order p. If $p-1$ vertices in $V(D)$ have degrees at least p, then D contains a Hamiltonian bypass.*

Acknowledgements

The author would like to thank the referees of the paper for careful reading and many helpful remarks.

References

[1] J. Bang-Jensen, G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, 2001.

[2] J.-C. Bermond and C. Thomassen, "Cycles in Digraphs-A Survey", *Journal of Graph Theory*, vol. 5, pp. 1-43, 1981.

[3] D. Kühn and D. Osthus, "A survey on Hamilton cycles in directed graphs", *European Journal of Combinatorics*, vol. 33, pp. 750-766, 2012.

[4] A. Benhocine and A.P. Wojda, "Bypasses in Digraphs", *Ars Combinatoria*, vol. 16, pp. 85-94, 1983.

[5] A. Benhocine, "On the existence of a specified cycle in digraphs with constraints on degrees", *Journal of Graph Theory*, vol. 8, pp. 101-107, 1984.

[6] S.Kh. Darbinyan, "On Hamiltonian bypasses in digraphs satisfying Meyniel-like condition", *Transactions of IIAP of NAS RA, Mathematical Problems of Computer Science*, vol. 20, pp. 7-19, 1998.

[7] S.Kh. Darbinyan, "On the specified cycles in oriented graphs", *Akademy Nauk Armyan. SSR Dokllady*, vol. 84, no. 2, pp. 51-55, 1987 (in Russian).

[8] S.Kh. Darbinyan and I.A. Karapetyan, "On Hamiltonian bypasses in one class of Hamiltonian digraphs", *Mathematical Problems of Computer Science*, vol. 41, pp. 23-37, 2014.

[9] S.Kh. Darbinyan, "On Hamiltonian bypasses in digraphs with the condition of Y. Manoussakis", *"2015 Computer Science and Information Technologies (CSIT)*, Yerevan, doi:10.1109/CSITechnol.2015.7358250, pp. 53-63, 2015.

[10] S.Kh. Darbinyan, "On Hmiltonian and strongly Hamilton-connected digraphs", *Akademy Nauk Armyan. SSR Doklady*, vol. 91, no. 1, pp. 3-6, 1990.(arXiv.1801.05166v1).

[11] S.Kh. Darbinyan, "A sufficient condition for a digraph to be Hamiltonian", *Akademy Nauk Armyan. SSR Doklady*,(in Russian), vol. 91, no. 2, pp. 57-59, 1990.

[12] R. Häggkvist and C. Thomassen, "On pancyclic digraphs", *Journal Combinatorial Theory ser. B*, vol. 20, pp. 20-40, 1976.

[13] J.A. Bondy and C. Thomassen, "A short proof of Meyniel's theorem", *Discrete Mathematics*, vol. 19, pp. 195-197, 1977.

[14] J. Bang-Jensen, G. Gutin and H. Li, "Sufficient conditions for a digraph to be Hamiltonian", *Journal of Graph Theory*, vol. 22, no. 2, pp. 181-187, 1996.

Submitted 03.08.2020, accepted 01.12.2020.

Մեկ նկատառում մեծ աստիճաններով կողմնորոշված գրաֆներում համիլտոնյան շրջանցումների մասին

Սամվել Խ. Դարբինյան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ
e-mail: samdarbin@iiap.sci.am

Ամփոփում

Ներկա աշխատանքում ապացուցվել է հետևյալ թեորեմը:

Թեորեմ: Դիցուք D -ն 2-ուժեղ կապակցված p-գագաթանի կողմնորոշված գրաֆ է, որի $p-1$ գագաթների աստիճանները փոքր չեն p թվից: Եթե D -ն համիլտոնյան է կամ D-ի փոքրագույն աստիճանը մեծ է $2(p-1)/5$ թվից, ապա այդ գրաֆը պարունակում է համիլտոնյան շրջանցում:

Բանալի բառեր՝ Կողմնորոշված գրաֆ, ցիկլ, համիլտոնյան ցիկլ, համիլտոնյան շրջանցում:

Одна заметка о гамильтоновых обходах в орграфах с большими степеньями

Самвел Х. Дарбинян

Институт проблем информатики и автоматизации НАН РА
e-mail: samdarbin@iiap.sci.am

Аннотация

В настоящей работе доказана следующая теорема:

Теорема: Пусть D есть 2-сильно связный p-вершинный орграф, в котором $p-1$ вершины имеют степень не меньше чем p. Если D гамильтонов или имеет минимальную степень больше чем $2(p-1)/5$, то D содержит гамильтонов обход.

Ключевые слова: Орграф, цикл, гамильтонов цикл, гамильтонов обход.