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Non-hamiltonian Graphs with Given Toughness

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Abstract

In 1973, Chvátal introduced the concept of toughness τ of a graph and conjectured that there exists a finite constant t_0 such that every t_0-tough graph (that is $\tau \geq t_0$) is hamiltonian. To solve this challenging problem, all efforts are directed towards constructing non-hamiltonian graphs with toughness as large as possible. The last result in this direction is due to Bauer, Broersma and Veldman, which states that for each positive ϵ, there exists a non-hamiltonian graph with $\frac{9}{4} - \epsilon \leq \tau < \frac{9}{4}$. The following related broad-scale problem, reminding the well-known pancyclicity or hypohamiltonicity, arises naturally: whedher there exists a non-hamiltonian graph with a given toughness. We conjecture that if there exist a non-hamiltonian t-tough graph then for each rational number a with $0 < a \leq t$ there exists a non-hamiltonian graph whose toughness is exactly a. In this paper we prove this conjecture for $t = \frac{9}{4} - \epsilon$ by using a number of additional modified building blocks to construct the required graphs.

Keywords: Hamilton cycle, Toughness of graph.

1. Introduction

Only finite undirected graphs without loops or multiple edges are considered. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges - by $E(G)$. The order and the independence number of G are denoted by n and α, respectively. For S a subset of $V(G)$, we denote by $G\backslash S$ the subgraph of G induced by $V(G)\backslash S$. The neighborhood of a vertex $x \in V(G)$ is denoted by $N(x)$. A graph G is hamiltonian if G contains a Hamilton cycle, i.e. a cycle of length n. A good reference for any undefined terms is [5].

The concept of toughness of a graph was introduced in 1973 by Chvátal [6]. Let $\omega(G)$ denote the number of components of a graph G. A graph G is t-tough if $|S| \geq t\omega(G\backslash S)$ for every subset S of the vertex set $V(G)$ with $\omega(G\backslash S) > 1$. The toughness of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Much of the research on this subject have been inspired by the following conjecture due to Chvátal [6].

Conjecture 1: *There exists a finite constant t_0 such that every t_0-tough graph is hamiltonian.*

To solve this challenging problem, all efforts are directed towards constructing non-hamiltonian graphs with toughness as large as possible. In [6], Chvátal constructed an infinite family of non-hamiltonian graphs with $\tau = \frac{3}{2}$, and then Thomassen [[4], $p.132$] found

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non-hamiltonian graphs with $\tau > \frac{3}{2}$. Later Enomoto et al. [7] have found non-hamiltonian graphs with $\tau \geq 2 - \epsilon$ for each positive ϵ. The last result in this direction is due to Bauer, Broersma and Veldman [2] inspired by special constructions introduced in [1] and [3].

Theorem A: *For each positive $\epsilon > 0$, there exists a non-hamiltonian graph with $\frac{9}{4} - \epsilon < \tau < \frac{9}{4}$.*

The following related broad-scale problem, reminding the well-known pancyclicity or hypohamiltonicity, arises naturally.

Problem: Does there exist a non-hamiltonian graph with the given toughness?

The following metaconjecture seems reasonable .

Conjecture 2: *If there exists a non-hamiltonian t-tough graph then for each rational number a with $0 < a \leq t$ there exists a non-hamiltonian graph whose toughness is exactly a.*

In this paper we prove this conjecture for $t = \frac{9}{4} - \epsilon$.

Theorem 1: *For each rational number t with $0 < t < \frac{9}{4}$, there exists a non-hamiltonian graph G with $\tau(G) = t$.*

Theorem 1 provides also a complete background for further investigation towards finding non-hamiltonian graphs with toughness at least $\frac{9}{4}$.

2. Preliminaries

To prove Theorem 1, we need both new and old graph constructions.

Definition 1. Let $L^{(1)}$ be a graph obtained from $C_8 = w_1w_2...w_8w_1$ by adding the edges w_2w_4, w_4w_6, w_6w_8 and w_2w_8. Put $x = w_1$ and $y = w_5$. This is the well-known building block L used to obtain $(\frac{9}{4} - \epsilon)$-tough non-hamiltonian graphs (see Figure 1 in [2]).

In this paper we will use a number of additional modified building blocks.

Definition 2: *Let $L^{(2)}$ be the graph obtained from $L^{(1)}$ by deleting the edges w_1w_2, w_2w_8 and identifying w_2 with w_8.*

Definition 3: *Let $L^{(3)}$ be the graph obtained from $L^{(1)}$ by adding a new vertex w_9 and the edges w_4w_9, w_6w_9.*

Definition 4: *Let $L^{(4)}$ be the graph obtained from the triangle $w_1w_2w_3w_1$ by adding the vertices w_4, w_5 and the edges w_1w_4, w_3w_5. Put $x = w_4$ and $y = w_5$.*

Definition 5: *For each $L \in \{L^{(1)}, L^{(2)}\}$, define the graph $G(L, x, y, l, m)$ $(l, m \in N)$ as follows. Take m disjoint copies $L_1, L_2, ..., L_m$ of L, with x_i, y_i the vertices in L_i corresponding to the vertices x and y in L $(i = 1, 2, ..., m)$. Let F_m be the graph obtained from $L_1 \cup ... \cup L_m$ by adding all possible edges between the pairs of vertices in $x_1, ..., x_m, y_1, ..., y_m$. Let $T = K_l$ and let $G(L, x, y, l, m)$ be the join $T \vee F_m$ of T and F_m.*

The following can be checked easily.

Claim 1: *The vertices x and y are not connected by a Hamilton path of $L^{(i)}$ $(i = 1, 2, 3)$.*

The proof of the following result occurs in [1].

Claim 2: *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geq 2l + 1$ then $G(H, x, y, l, m)$ is non-hamiltonian.*

3. Proof of Theorem 1

By the definition, the toughness $\tau(G)$ is a rational number. Let t be any rational number with $0 < t < \frac{9}{4}$ and let $t = \frac{a}{b}$ for some integers a, b.

Case 1: $0 < \frac{a}{b} < 1$.

Let $K_{a,b}$ be the complete bipartite graph $G = (V_1, V_2; E)$ with vertex classes V_1 and V_2 such that $|V_1| = a$ and $|V_2| = b$. Since $\frac{a}{b} < 1$, we have $\alpha(G) = b > (a+b)/2$ and therefore, $K_{a,b}$ is a non-hamiltonian graph. Clearly, $\tau \leq |V_1|/\omega(G\backslash V_1) = a/b$. Choose $S \subset V(G)$ such that $\tau(G) = |S|/\omega(G\backslash S)$. Put $S \cap V_i = S_i$ and $|S_i| = s_i$ $(i = 1, 2)$. If $V_i\backslash S \neq \emptyset$ $(i = 1, 2)$ then clearly $\omega(G\backslash S) = 1$, which is impossible by the definition. Hence, $V_i\backslash S = \emptyset$ for some $i \in \{1, 2\}$.

Case 1.1: $i = 2$.

It follows that

$$\tau = \frac{b + s_1}{a - s_1} \geq \frac{b}{a}.$$

Recalling that $\tau \leq \frac{a}{b}$, we have $b^2 \leq a^2$, contradicting the hypothesis $\frac{a}{b} < 1$.

Case 1.2: $i = 1$.

It follows that

$$\tau = \frac{s_2 + a}{b - s_2} \geq \frac{a}{b},$$

implying immediately that $\tau = \frac{a}{b}$.

Case 2: $\frac{a}{b} = 1$.

Let G be a graph obtained from $C_6 = v_1v_2...v_6v_1$ by adding a new vertex v_7 and the edges v_1v_7, v_4v_7, v_2v_6. Clearly, G is not hamiltonian and $\tau(G) = 1$.

Case 3: $1 < \frac{a}{b} < \frac{3}{2}$.

Case 3.1: $\frac{a}{b} < \frac{3}{2} - \frac{1}{b}$.

Let V_1, V_2, V_3 be pairwise disjoint sets of vertices:

$$V_1 = \{x_1, x_2, ..., x_{a-b+1}\},\ V_2 = \{y_1, y_2, ..., y_b\},\ V_3 = \{z_1, z_2, ..., z_b\}.$$

Join each x_i to all the other vertices and each z_i to every other z_j as well as to the vertex y_i with the same subscript i. Call the resulting graph G. Choose $W \subset V(G)$ such that $\tau(G) = |W|/\omega(G\backslash W)$. Put $m = |W \cap V_3|$. Clearly, W is a minimal set whose removal from G results in a graph with $\omega(G\backslash W)$ components. As W is a cutset, we have $V_1 \subset W$ and $m \geq 1$. From the minimality of W we easily conclude that $V_2 \cap W = \emptyset$ and $m \leq b-1$. Then we have $|W| = m + a - b + 1$ and $\omega(G\backslash W) = m + 1$. Hence,

$$\tau(G) = \frac{|W|}{\omega(G\backslash W)} = \min_{1 \leq m \leq b-1} \frac{m + a - b + 1}{m + 1} = \frac{a}{b}.$$

To see that G is non-hamiltonian, let us assume the contrary, i.e. let C be a Hamilton cycle in G. Denote by F the set of edges of C having at least one endvertex in V_2. Since V_2 is independent, we have $|F| = 2|V_2|$. On the other hand, there are at most $2|V_1|$ edges in F having one endvertex in V_1 and at most $|V_3|$ edges in F having one endvertex in V_3. Thus,

$$2b = 2|V_2| = |F| \leq 2|V_1| + |V_3| = 2(a - b + 1) + b = 2a - b + 2.$$

But this is equivalent to $a/b \geq 3/2 - 1/b$, contradicting the hypothesis.

Case 3.2: $\frac{a}{b} \geq \frac{3}{2} - \frac{1}{b}$.

By choosing a sufficiently large $q \in N$ with

$$\frac{a}{b} = \frac{aq}{bq} < \frac{3}{2} - \frac{1}{bq},$$

we can argue as in Case 3.1.

Case 4: $\frac{a}{b} = \frac{3}{2}$.

An example of a non-hamiltonian graph with $\tau = 3/2$ is obtained when in the Petersen graph, each vertex is replaced by a triangle.

Case 5: $\frac{3}{2} < \frac{a}{b} < \frac{7}{4}$.

Claim 3: *For $l \geq 2$ and $m \geq 1$,*

$$\tau\left(G\left(L^{(2)}, x, y, l, m\right)\right) = \frac{l+3m}{1+2m}.$$

Proof. Let $G = G(L^{(2)}, x, y, l, m)$ for some $l \geq 2$ and $m \geq 1$. Choose $S \subseteq V(G)$ such that

$$\omega(G \backslash S) > 1, \quad \tau(G) = \frac{|S|}{\omega(G \backslash S)}.$$

Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \backslash S_i$ that contain neither x_i nor y_i $(i = 1, ..., m)$. Then

$$\tau(G) = \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + \sum_{i=1}^m s_i}{1 + \sum_{i=1}^m \omega_i},$$

where

$$c = \begin{cases} 0 & \text{if} \quad x_i, y_i \in S \text{ for all } i \in \{1, ..., m\} \\ 1 & \qquad \text{othrwise.} \end{cases}$$

It is easy to see that

$$\omega_i \leq 2, \quad s_i \geq \frac{3}{2}\omega_i \quad (i = 1, ..., m).$$

Then

$$\tau \geq \frac{l + \frac{3}{2}\sum_{i=1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i} = \frac{l - \frac{3}{2}}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2}$$
$$\geq \frac{l - \frac{3}{2}}{1 + 2m} + \frac{3}{2} = \frac{l+3m}{1+2m}.$$

Set $U = V(T) \cup U_1 \cup ... \cup U_m$, where U_i is the set of vertices of L_i with the degree at least 4 in L_i $(i = 1, ..., m)$. The proof of Claim 3 is completed by observing that

$$\tau(G) \leq \frac{|U|}{\omega(G \backslash U)} = \frac{l+3m}{2m+1}. \qquad \blacksquare$$

Case 5.1:$b = 2k + 1$ *for some integer k.*

Consider the graph $G(L^{(2)}, x, y, a - \frac{3}{2}(b-1), \frac{b-1}{2})$.

Case 5.1.1: $\frac{a}{b} \leq \frac{7}{4} - \frac{9}{4b}$.

By the hypothesis,

$$m = \frac{b-1}{2} \geq 2\left(a - \frac{3}{2}(b-1)\right) + 1 = 2l + 1.$$

By Claim 2, G is not hamiltonian. Clearly $b \geq 3$, implying that $m = (b-1)/2 \geq 1$. If $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$ then $l = a - \frac{3}{2}(b-1) \geq 2$ and by Claim 3, $\tau(G) = \frac{a}{b}$. Now let $\frac{a}{b} < \frac{3}{2} + \frac{1}{2b}$. By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \geq \frac{3}{2} + \frac{1}{2bq},$$

we can argue as in the previous case.

Case 5.1.2: $\frac{a}{b} > \frac{7}{4} - \frac{9}{4b}$.

By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \leq \frac{7}{4} - \frac{9}{4bq},$$

we can argue as in Case 5.1.1.

Case 5.2: *$b = 2k$ for some integer k.*

Consider the graph G' obtained from $G(L^{(2)}, x, y, l, m)$ by replacing L_m with $L^{(3)}$.

Claim 4: *For $l \geq 2$ and $m \geq 1$,*

$$\tau(G') = \frac{l + 3m + 1}{2(m+1)}.$$

Proof: Choose $S \subseteq V(G')$ such that $\omega(G' \backslash S) > 1$ and $\tau(G') = |S|/\omega(G' \backslash S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \backslash S_i$ that contain neither x_i nor y_i $(i = 1, ..., m)$. Since $s_i \geq \frac{3}{2}\omega_i$ $(i = 1, ..., m-1)$ and $s_m \geq \frac{4}{3}\omega_m$, we have

$$\tau(G') \geq \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + \frac{3}{2}\sum_{i=1}^{m-1} \omega_i + \frac{4}{3}\omega_m}{1 + \sum_{i=1}^m \omega_i} = \frac{l - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, ..., m\}$ and $c = 1$, otherwise. Observing also that $\omega_i \leq 2$ $(i = 1, ..., m-1)$ and $\omega_m \leq 3$, we obtain

$$(l-2)\sum_{i=1}^m \omega_i + \frac{1}{3}(m+1)\omega_m \leq (l-2)(2m+1) + (m+1) \leq 2l(m+1).$$

But this is equivalent to

$$\frac{l - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2} \geq \frac{l-2}{2(m+1)} + \frac{3}{2},$$

implying that

$$\tau(G') \geq \frac{l-2}{2(m+1)} + \frac{3}{2} = \frac{l + 3m + 1}{2(m+1)}.$$

Set $U = V(T) \cup U_1 \cup ... \cup U_m$, where U_i is the set of vertices of L_i with the degree at least 4 in L_i $(i = 1, ..., m)$. The proof of Claim 4 is completed by observing that

$$\tau(G') \leq \frac{|U|}{\omega(G \backslash U)} = \frac{l + 3m + 1}{2(m+1)}. \quad \blacksquare$$

Consider the graph G' with $m = \frac{b}{2} - 1$ and $l = a - \frac{3}{2}b + 2$. Clearly $m = \frac{b}{2} - 1 \geq 1$ and $l = a - \frac{3}{2}b + 2 \geq 2$. By Claim 4, $\tau(G') = \frac{a}{b}$. If $\frac{a}{b} \leq \frac{7}{4} - \frac{3}{b}$ then $m \geq 2l + 1$, and by Claim 2, G' is not hamiltonian. Otherwise, by choosing a sufficiently large q with

$$\frac{a}{b} = \frac{aq}{bq} \leq \frac{7}{4} - \frac{3}{b},$$

we can argue as in the previous case.

Case 6: $\frac{7}{4} - \epsilon < \frac{a}{b} \leq 2$.

Let $m = m_1 + m_2 \geq 2l + 1$ and let G'' be the graph obtained from $G(L^{(1)}, x, y, l, m)$ by replacing L_i with $L^{(2)}$ $(i = m_1 + 1, m_1 + 2, ..., m)$. By Claim 2, G'' is not hamiltonian.

Claim 5: *For $l \geq 2$, $m \geq 1$ and $m_2 \geq l - 2$,*

$$\tau(G'') = \frac{l + 3m_2}{2m_2 + 1}.$$

Proof: Choose $S \subseteq V(G'')$ such that $\tau(G'') = |S|/\omega(G''\backslash S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \backslash S_i$ that contain neither x_i nor y_i $(i = 1, ..., m)$. Since $s_i \geq 2\omega_i$ $(i = 1, ..., m_1)$ and $s_i \geq \frac{3}{2}\omega_i$ $(i = m_1 + 1, ..., m)$, we have

$$\tau(G'') \geq \frac{l + \sum_{i=1}^{m_1} s_i + \sum_{i=m_1+1}^{m} s_i}{c + \sum_{i=1}^{m} \omega_i} \geq \frac{l + 2\sum_{i=1}^{m_1} \omega_i + \frac{3}{2}\sum_{i=m_1+1}^{m} \omega_i}{1 + \sum_{i=1}^{m} \omega_i}$$

$$\frac{l + \frac{1}{2}\sum_{i=1}^{m_1} \omega_i - \frac{3}{2} + \frac{3}{2}(1 + \sum_{i=1}^{m} \omega_i)}{1 + \sum_{i=1}^{m} \omega_i} = \frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2(1 + \sum_{i=1}^{m} \omega_i)} + \frac{3}{2},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, ..., m\}$ and $c = 1$, otherwise. Observing that $\omega_i \leq 2$ $(i = 1, ..., m)$, we obtain

$$(2l - 3)\sum_{i=m_1+1}^{m} \omega_i - (2m_2 - 2l + 4)\sum_{i=1}^{m_1} \omega_i \leq 4lm_2 - 6m_2.$$

But this is equivalent to

$$\frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2\,(1 + \sum_{i=1}^{m} \omega_i)} + \frac{3}{2} \geq \frac{2l - 3}{2(2m_2 + 1)} + \frac{3}{2},$$

implying that

$$\tau(G'') \geq \frac{2l - 3}{2(2m_2 + 1)} + \frac{3}{2} = \frac{l + 3m_2}{2m_2 + 1}.$$

Set $U = V(T) \cup U_1 \cup ... \cup U_m$, where U_i is the set of vertices of L_i with the degree at least 4 in L_i $(i = 1, ..., m)$. The proof of Claim 5 is completed by observing that

$$\tau(G'') \leq \frac{|U|}{\omega(G\backslash U)} = \frac{l + 3m_2}{2m_2 + 1}. \quad \blacksquare$$

Case 6.1: $b = 2k + 1$ *for some integer* k.
Consider the graph G'' with $m_2 = \frac{b-1}{2}$ and $l = a - \frac{3}{2}(b - 1)$.

Case 6.1.1: $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$.
Since $\frac{a}{b} \leq 2$, we have

$$m_2 = \frac{b - 1}{2} \geq a - \frac{3}{2}(b - 1) - 2 = l - 2.$$

Next, since $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$, we have $l = a - \frac{3}{2}(b - 1) \geq 2$. By Claim 5, $\tau(G'') = \frac{a}{b}$.

Case 6.1.2: $\frac{a}{b} < \frac{3}{2} + \frac{1}{2b}$.
By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \geq \frac{3}{2} + \frac{1}{2bq},$$

we can argue as in Case 6.1.1.

Case 6.2: $b = 2k$ *for some integer* k.

Consider the graph G''' obtained from G'' by replacing L_m with $L^{(3)}$.

Claim 6: For $l \geq 2$, $m \geq 1$ and $m_2 \geq l - 2$,

$$\tau(G''') = \frac{l + 3m_2 + 1}{2(m_2 + 1)}.$$

Proof: Choose $S \subseteq V(G''')$ such that $\tau(G''') = |S|/\omega(G'''\backslash S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \backslash S_i$ that contain neither x_i nor y_i $(i = 1, ..., m)$. Since $s_i \geq 2\omega_i$ $(i = 1, ..., m_1)$, $s_i \geq \frac{3}{2}\omega_i$ $(i = m_1 + 1, ..., m - 1)$ and $s_m \geq \frac{4}{3}\omega_m$, we have

$$\tau(G''') \geq \frac{l + \sum_{i=1}^{m_1} s_i + \sum_{i=m_1+1}^{m-1} s_i + s_m}{c + \sum_{i=1}^{m} \omega_i}$$

$$\geq \frac{l + 2\sum_{i=1}^{m_1} \omega_i + \frac{3}{2}\sum_{i=m_1+1}^{m-1} \omega_i + \frac{4}{3}\omega_m}{1 + \sum_{i=1}^{m} \omega_i}$$

$$= \frac{l + \frac{1}{2}\sum_{i=1}^{m_1} \omega_i - \frac{1}{6}\omega_m + (\frac{3}{2}\sum_{i=1}^{m_1} \omega_i + \frac{3}{2}\sum_{i=m_1+1}^{m})}{1 + \sum_{i=1}^{m} \omega_i}$$

$$= \frac{l + \frac{1}{2}\sum_{i=1}^{m_1} \omega_i - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} + \frac{3}{2},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, ..., m\}$ and $c = 1$, otherwise. Observing that $\omega_i \leq 2$ $(i = 1, ..., m - 1)$ and $\omega_m \leq 3$, we obtain

$$(l - 2)\sum_{i=m_1+1}^{m} \omega_i + \frac{1}{3}(m_2 + 1)\omega_m - (m_2 - l + 3)\sum_{i=1}^{m_1} \omega_i \leq l + 2lm_2 + 2.$$

But this is equivalent to

$$\frac{l + \frac{1}{2}\sum_{i=1}^{m_1} \omega_i - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} + \frac{3}{2} \geq \frac{l - 2}{2(m_2 + 1)} + \frac{3}{2},$$

implying that

$$\tau(G''') \geq \frac{l - 2}{2(m_2 + 1)} + \frac{3}{2} = \frac{l + 3m_2 + 1}{2(m_2 + 1)}.$$

Set $U = V(T) \cup U_1 \cup ... \cup U_m$, where U_i is the set of vertices of L_i with the degree at least 4 in L_i $(i = 1, ..., m)$. The proof of Claim 6 is completed by observing that

$$\tau(G''') \leq \frac{|U|}{\omega(G\backslash U)} = \frac{l + 3m_2 + 1}{2(m_2 + 1)}. \qquad \blacksquare$$

Consider the graph G''' with $m_2 = \frac{b}{2} - 1$ and $l = a - \frac{3}{2}b + 2$.

Case 6.2.1: $\frac{a}{b} \leq 2 - \frac{1}{b}$.

By the hypothesis, $m_2 = \frac{b}{2} - 1 \geq \left(a - \frac{3}{2}b + 2\right) - 2 = l - 2$. Next, since $\frac{a}{b} > \frac{7}{4} - \epsilon > \frac{3}{2}$, we have $l = \frac{3}{2}b + 2 \geq 2$. By Claim 6, $\tau(G''') = \frac{a}{b}$.

Case 6.2.2: $\frac{a}{b} > 2 - \frac{1}{b}$.

By choosing a sufficiently large integer q with $\frac{a}{b} = \frac{aq}{bq} \leq 2 - \frac{1}{bq}$, we can argue as in Case 6.2.1.

Case 7: $2 < \frac{a}{b} < \frac{9}{4}$.

Case 7.1: $b = 2k+1$ *for some integer* k.

Case 7.1.1: $\frac{a}{b} \leq \frac{9}{4} - \frac{11}{4b}$.

Take the graph $G\left(L^{(1)}, x, y, a-2b+2, \frac{b-1}{2}\right)$. Since $\frac{a}{b} > 2$, we have $l = a - 2b + 2 \geq 2$. Next, the hypothesis $\frac{a}{b} \leq \frac{9}{4} - \frac{11}{4b}$ is equivalent to

$$m = \frac{b-1}{2} \geq 2(a-2b+2)+1 = 2l+1.$$

By Claim 1, $G\left(L^{(1)}, x, y, a-2b+2, \frac{b-1}{2}\right)$ is not hamiltonian. The toughness $\tau\left(G\left(L^{(1)}, x, y, a-2b+2, \frac{b-1}{2}\right)\right)$ can be determined exactly as in the proof of Theorem A [2],

$$\tau\left(G\left(L^{(1)}, x, y, a-2b+2, \frac{b-1}{2}\right)\right) \geq \frac{l+4m}{2m+1} = \frac{a}{b}.$$

Case 7.1.2: $\frac{a}{b} > \frac{9}{4} - \frac{11}{4b}$.

By choosing a sufficiently large integer q with

$$\frac{aq}{bq} = \frac{a}{b} \leq \frac{9}{4} - \frac{11}{4bq},$$

we can argue as in Case 7.1.1.

Case 7.2: $b = 2k$ *for some positive integer* k.

Take the graph G'''' obtained from $G\left(L^{(1)}, x, y, a-2b+2, \frac{b}{2}\right)$ by replacing L_m with $L^{(4)}$. Since $\frac{a}{b} > 2$, we have $l = a-2b+2 > 2$. We have also $m = \frac{b}{2} > 1$, since $b \geq 3$.

Claim 7: *For* $l \geq 2$ *and* $m \geq 1$,

$$\tau(G'''') = \frac{l+4m-2}{2m}.$$

Proof: Choose $S \subseteq V(G''')$ such that $\tau(G''') = |S|/\omega(G'''\backslash S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \backslash S_i$ that contain neither x_i nor y_i $(i = 1, ..., m)$. Since $s_i \geq 2\omega_i$ $(i = 1, ..., m)$, $\omega_i \leq 2$ $(i = 1, ..., m-1)$ and $\omega_m \leq 1$, we have

$$\tau(G'''') = \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + 2\sum_{i=1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i}$$
$$= \frac{l-2}{1+\sum_{i=1}^m \omega_i} + 2 \geq \frac{l-2}{2m} + 2 = \frac{l+4m-2}{2m},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, ..., m\}$ and $c = 1$, otherwise. Set $U = V(T) \cup U_1 \cup ... \cup U_m$, where U_i is the set of vertices of L_i with the degree at least 4 in L_i $(i = 1, ..., m)$. The proof of Claim 7 is completed by observing that

$$\tau(G'''') \leq \frac{|U|}{\omega(G\backslash U)} = \frac{l+4m-2}{2m}. \qquad \blacksquare$$

Case 7.2.1: $\frac{a}{b} \leq \frac{9}{4} - \frac{3}{b}$.

By the hypothesis,

$$m-1=\frac{b}{2}-1\geq 2(a-2b+2)+1=2l+1.$$

By Claim 2, G'''' is not hamiltonian and by Claim 7, $\tau\left(G''''\right)=\frac{a}{b}$.

Case 7.2.2: $\frac{a}{b}>\frac{9}{4}-\frac{3}{b}$.

By choosing a sufficiently large integer q with

$$\frac{aq}{bq}=\frac{a}{b}\leq\frac{9}{4}-\frac{3}{3bq},$$

we can argue as in Case 7.2.1. Theorem 1 is proved. ■

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Տրված կոշտությամբ ոչ համիլտոնյան գրաֆներ

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Ամփոփում

Գրաֆի կոշտության τ բնութագրիչը ներմուծել է Խվատալը 1973-ին: Ըստ Խվատալի վարկածի, գոյություն ունի այնպիսի t_0 վերջավոր թիվ, որ կամայական t_0-կոշտ գրաֆ (ինչը նշանակում է, որ $\tau\geq t_0$) համիլտոնյան է: Այս մարտահրավերային խնդրի լուծման բոլոր ջանքերը և տեխնիկան ուղղված են առավելագույն կոշտություն ունեցող ոչ համիլտոնյան գրաֆների կառուցմանը, անտեսելով ավելի ցածր կոշտության գրաֆները: Այս ուղղությամբ ստացված վերջին արդյունքը, որը ստացել են Բաուերը, Բրոերսման և Վելդմանը, պնդում է, որ կամայական դրական ε թվի համար գոյություն ունի ոչ համիլտոնյան գրաֆ, որի կոշտությունը գտնվում է $\frac{9}{4}-\varepsilon\leq\tau<\frac{9}{4}$ սահմաններում: Հետևյալ ավելի ընդգրկուն խնդիրը իր դրվածքով հիշեցնում է պանցիկլիկության և հիպոհամիլտոնյան գրաֆների գոյության խնդիրները. գոյություն ունի, արդյոք ոչ համիլտոնյան գրաֆ տրված կոշտությամբ: Ըստ մեր վարկածի, եթե գոյություն ունի ոչ համիլտոնյան t-կոշտ գրաֆ, ապա $(0,t]$ ինտերվալին պատկանող կամայական a ռացիոնալ թվի համար գոյություն ունի ոչ համիլտոնյան գրաֆ, որի կոշտությունը ուղիղ է: Ներկա աշխատանքում այս վարկածը ապացուցվում է $(0,\frac{9}{4})$ միջակայքին պատկանող կամայական ռացիոնալ թվի համար: Պահանջված գրաֆները կառուցելու համար օգտագործվել են մի շարք նոր կառուցվածքային միավորներ, որոնք տարբերվում են գրականության մեջ հայտնի միավորներից:

Негамильтоновые графы с заданной жесткостью

Ж. Никогосян

Аннотация

Понятие жесткости $\tau(G)$ графа G было введено Хваталом в 1973 году. По известной гипотезе Хватала, существует конечное число t_0 такое, что каждый t_0-жесткий граф (это означает, что $\tau \geq t_0$) гамильтонов. Для решения этой стимулирующей проблемы все усилия концентрировались на построение негамильтоновых графов с максимальной жесткостью, не уделяя внимания на графы с низкой жесткостью. Последний результат в этом направлении, который получили Бауер, Броерсма и Вельдман, утверждает, что для каждого положительного числа ε, существует негамильтонов граф с жесткостью $\frac{9}{4}-\varepsilon \leq \tau < \frac{9}{4}$. Подобно задачам панцикличности и существования гипогамильтоновых графов, мы рассматриваем более емкую задачу: существует ли негамильтонов граф с заданной жесткостью. По нашей гипотезе, если существует негамильтонов t-жесткий граф, то для любого рационального числа a из интервала $(0,t]$, существует негамильтонов граф, жесткость которого равна a. В данной работе мы доказываем эту гипотезу для любого рационального числа из интервала $(0,\frac{9}{4})$. Для построения требуемых графов были использованы некоторые новые дополнительные конструктивные блоки.