Mathematical Problems of Computer Science 40, 5–12, 2013.

Method of Local Interchange to Investigate Gossip Problems

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Abstract

In this paper the method of *local interchange* is introduced to investigate gossip problems. This method is based on a repetitive use of *permute higher* and *permute lower* operations, which map one gossip graph with n vertices to another by moving only its edges without changing the labels of edges (the moments of corresponding calls). Using this method we obtain minimum time gossip graphs in which no one hears his own information (NOHO graphs) from gossip graphs based on Knödel graphs. The value of minimal time is $T = \lceil \log n \rceil$. This method also allowed us to find an alternative way of the proof that the number of minimal necessary calls in gossip schemes is $2n-4$, $n \geq 4$.

Keywords: Graphs, Networks, Telephone problem, Gossip problem.

1. Introduction

Gossiping is one of the basic problems of information dissemination in communication networks. The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see ex. [4] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). Consider a set of n persons (nodes) each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with minimum length (minimal gossip scheme), by which all the nodes will know all pieces of information (complete gossiping). It has been shown in numerous works [1]-[4] that the minimal number of calls is $2n-4$, when $n \geq 4$ and $1, 3$, for $n = 2, 3$, respectively. Since then many variations of gossip problem have been introduced and investigated [5]–[10], [13]–[17].

Another variant of the Gossip problem can be formulated by considering the minimum amount of time required to complete gossiping among n persons, where the calls between the non-overlapping pairs of nodes can take place simultaneously and each call requires one unit of time [11, 12, 18].

In section 2 we introduce the method of *local interchange*, which is based on a repetitive use of *permute higher* and *permute lower* operations. These operations locally act on the adjacent edges to a given edge in a gossip graph (a non-local *interchange of two vertices from* *some point onwards in a list of telephone calls* is defined in [4]). In section 3 the method of *local interchange* is demonstrated for the alternative derivation of the minimum number of calls in gossip scheme. Another application of *local interchange* is described in section 4, where the procedure for obtaining minimum time gossip graphs in which no one hears his own information (NOHO graphs, [20, 22]) from gossip graphs based on Knödel graphs is found. Since the minimal time of gossip graphs is known [18], this procedure allowed us to establish the value of minimal time for NOHO graphs $T = \lceil \log n \rceil$.

2. The Method

A gossip scheme (a sequence of calls between n nodes) can be represented by an undirected edge-labeled graph $G = (V, E)$ with $|V(G)| = n$ vertices. The vertices and edges of G represent correspondingly the nodes and the calls between the pairs of nodes of a gossip scheme. Such graphs may have multiple edges, but not self loops. An edge-labeling of G is a mapping $t_G : E(G) \to Z^+$. The label $t_G(e)$ of a given edge $e \in E(G)$ represents the moment of time, when the corresponding call occurs.

A sequence $L = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$ with vertices $v_i \in V(G)$ for $0 \le i \le k$ and edges $e_i \in E(G)$ for $1 \le i \le k$ is called a walk of length k from a vertex v_0 to a vertex v_k in G, if each edge e_i joins two vertices v_{i-1} and v_i for $1 \le i \le k$. A walk, in which all the vertices are distinct, is called a path. If $t_G(e_i) < t_G(e_j)$ for $1 \le i < j \le k$, then L is an ascending path from v_0 to v_k in G. Given two vertices u and v, if there is an ascending path from u to v, then v receives the information of u.

Note that two non-adjacent edges can have the same label. Since we consider only (strictly) ascending paths, then such edges (i.e. calls) are independent, which means that during the consideration of minimal number of calls the edges with the same label can be reordered arbitrarily but for any $t_1 < t_2$ all the edges with the label t_1 are ordered before any of the edges with the label t_2. Therefore, in this case we can assume that there are not any two edges e_1 and e_2 such that $t_G(e_1) = t_G(e_2)$.

Denote the set of edges adjacent to a given vertex v by $E_v(G)$. Given an edge e and one of its two endpoints v, we consider the following two subsets of the set $E_v(G)$:

$$\rho_v^+(e, G) = \{e' \in E_v(G) | t_G(e') \ge t_G(e)\}, \quad (1)$$

$$\rho_v^-(e, G) = \{e' \in E_v(G) | t_G(e') \le t_G(e)\}. \quad (2)$$

Sometimes we omit the argument G in notations.

Definition 1: *An identification of two vertices v_1 and v_2 [4] in a gossip graph G is a gossip graph G', which is defined as follows: The edges between the vertices v_1 and v_2 are deleted and these two vertices are replaced by a vertex u, whose set of incident edges is $E_u(G') = E_{v_1}(G) \cup E_{v_2}(G)$*

An interchange of two vertices in a calling scheme is defined as follows: started from the indicated moment of a time to the end of the calling scheme two vertices (and the edges adjacent to them) are replaced by each other. In this paper we define the concept of local interchange, i.e. an interchange that is defined only for adjacent vertices.

Definition 2: *The permute higher operation $P^+(e)$ on a selected edge $e \in E(G)$ connecting vertices u and v is called a modification of G, which moves the edges of G adjacent to e as* *follows:*

$$E_u(P^+(e)G) = \rho_u^-(e,G) \cup \rho_v^+(e,G), \tag{3}$$

and

$$E_v(P^+(e)G) = \rho_v^-(e,G) \cup \rho_u^+(e,G). \tag{4}$$

Correspondingly, the permute lower operation $P^-(e)$ moves the edges of G adjacent to e as follows:

$$E_u(P^-(e)G) = \rho_u^+(e,G) \cup \rho_v^-(e,G), \tag{5}$$

and

$$E_v(P^-(e)G) = \rho_v^+(e,G) \cup \rho_u^-(e,G). \tag{6}$$

The operators P^+ and P^- are called the *operators of local interchange.*

Lemma 1: *The result of the action of the operators of local interchange on a complete gossip graph is also a complete gossip graph.*

Actually, if a call between two participants takes place at the moment t_0, started from that point they both have the same information and are equivalent. Hence, if we change all calls which took place after that moment (permute higher) our new gossip scheme would also be complete. At the same time, we changed neither the number of edges, nor the number of rounds required to perform gossiping. The same assumptions could be made in case of permute lower operation.

Let us define an operation on gossip graphs $A^+ = P^+(e_1)P^+(e_2)\dots P^+(e_p)$ ($A^- = P^-(e_1)P^-(e_2)\dots P^-(e_p)$) is the sequence of permute higher (lower) operations on edges e_i, $i = 1,\dots,p$.

Lemma 2: *The result of the operation A^+ (A^-) does not depend on the order of the edges e_i.*

Proof. Without loss of generality we can only consider the permute higher operation A^+ with $p = 2$. Assume that $t_G(e_1) < t_G(e_2)$, where t_G is edge-ordering of gossip graph G. There are two possible cases:

1) Edges e_1 and e_2 are adjacent. In this case, if $A^+ = P^+(e_1)P^+(e_2)$, after $P^+(e_1)$ all the edges which are adjacent to e_1 (including e_2) will change their endpoint from one endpoint of e_1 to another and after $P^+(e_2)$ all the edges for which $t_G(e) > t_G(e_2)$ also will be permuted. If $A^+ = P^+(e_2)P^+(e_1)$, then after the first phase of A^+ all the edges with $t_G(e) > t_G(e_2)$ will be permuted and after the second phase all the edges which are adjacent to e_1 and have $t_G(e) > t_G(e_1)$ (including e_2) will also be permuted. These operations are demonstrated in Fig. 1. It is obvious that the result of these two variations of A^+ is the same.

2) Edges e_1 and e_2 are not adjacent. This is a simpler case in which the operation $P^+(e)$ on one of them does not affect directly the other. So, it is obvious that all the variations of A^+ are equivalent.

■

So, in this section the operations of local interchange and their main properties were described. In the next two section we give some application of the operations described in this section.

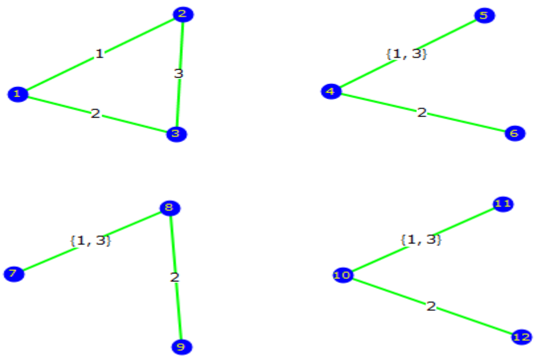

Fig. 1. Equivalence of different variations of A^+

3. Minimum Gossip Graphs

Theorem 1: *The minimum required number of calls in complete gossip schemes is* $2n-4$, $n \geq 4$.

Proof. As already mentioned in the introduction it has been shown in numerous works [1, 2, 3, 4] that the minimal number of calls in gossip schemes is $2n-4$, where n is the number of vertices. In this section we will prove this statement in a complectly new way, which is based on the operations of local interchange.

Let us denote by $f(n)$ the minimum number of edges required to construct a gossip graph on n vertices. There are many solutions of this problem that give the number of edges (calls) equal to $2n-4$. Therefore, $f(n) \leq 2n-4$ is valid. So, the important part is to prove that $f(n) \geq 2n-4$ also.

Consider a gossip graph with n vertices and the number of edges equal to $f(n)$. Our goal is to prove that $f(n) \geq f(n-1)+2$. After that, taking into account that $f(4)=4$ we will get that $f(n) \geq 2n-4$.

Suppose we have a graph G with $f(n)$ calls. There are 3 possible kinds of graphs depending on the repetition of information.

1) It contains a cycle. Suppose a $C=(e_1, e_2, \dots, e_p)$ is a cycle with length p and the edges e_1 and e_p are adjacent to the vertex v (see Fig. 2).

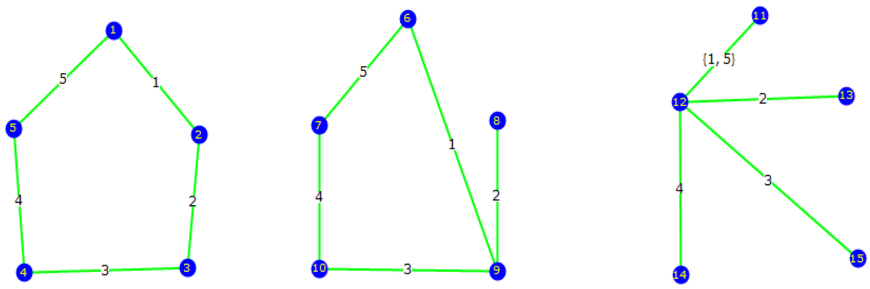

Fig. 2. Cycle C in gossip graphs

The simplest case is if $p=2$. If this holds, we can apply an operation of the identification of two vertices connected by this multiple edge which is described in the section above. Obviously, after this operation on complete gossip graphs, the obtained new gossip graphs are also complete (it does not affect on ascending paths between the pairs of vertices). Hence, our new graph G' with $n-1$ vertices has $|E(G')| \geq f(n-1)$ edges. On the other hand G' is obtained from G by removing two edges and merging two vertices, thus, $|E(G')| = f(n) - 2$. Hence, we obtain

$$f(n) \geq f(n-1) + 2. \tag{7}$$

Now consider the case of $p \neq 2$ and an operation $A^- = (P^-(e_2)P^-(e_3)\ldots P^-(e_{p-1}))$. As it is proved in the previous section the result of application of this operation on a complete gossip graph is also a complete gossip graph. In the result the edges e_1 and e_p will join and form a double-edge. Thus, we came to the case of $p = 2$. Further steps are described above.

2) There are no cycles (NOHO graphs). In this case we can refer to [6, 20], but let us also consider this case. Let $L_1 = (e_1, e_2, \ldots, e_n)$ and $L_2 = (l_1, l_2, \ldots, l_m)$ be ascending paths from the vertex v to the vertex u.

Let us define the operations

$$A_e^- = (P^-(e_2)P^-(e_3)\ldots P^-(e_{n-1})) \tag{8}$$

and

$$A_l^- = (P^-(l_2)P^-(l_3)\ldots P^-(l_{m-1})) \tag{9}$$

on gossip graph. After application of these operations we will obtain a tetragon which involves the vertices u and v and the edges e_1, l_1, e_n, l_m. Then we choose the greatest from the pair e_1, l_1 and apply operation "permute lower" on it. We will obtain a triangle that is also a cycle (case 1). This process is demonstrated in Fig. 3.

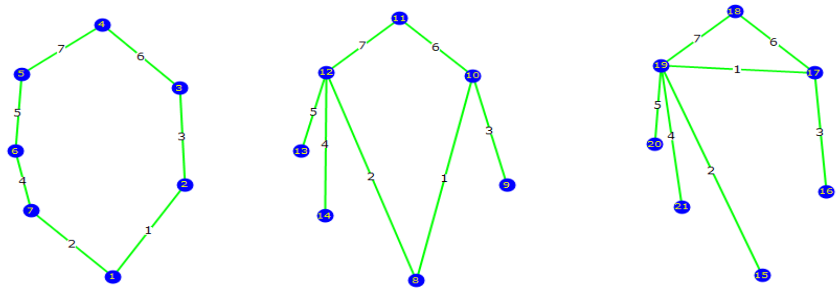

Fig. 3. Ascending paths in gossip graphs

3) There are no duplicates of information (NODUP graphs). In this case there is exactly one ascending path between any pair of vertices. It has been shown in [21], that in this case $f'(n) = 2.25n - 6$, $n = 4k$, $k \geq 2$, where $f'(n)$ is thi minimum number of edges of NODUP graph and it coincides with $2n - 4$ only when $n = 8$. So, if it is a NODUP graph, it always has more edges than $2n - 4$ and the only exception is the case when $n = 8$. Case of $n = 4$ was not considered in [21], but in this case $f(n)$ and $f'(n)$ also coincide.

■

In the next section other applications of local interchange operations are described.

4. NOHO Graphs Based on Knödel Graphs

In [19] it is mentioned that the minimum time needed to complete gossiping is $\lceil \log n \rceil$. In [22] a problem of finding minimum time of gossiping in case of NOHO graphs (problem 26) is raised. In this section we will show a new method of construction of NOHO graphs with minimum gossiping time by applying an operation of local interchange on Knödel graphs ([7]).

Definition 3: *The Knödel graph on $n \geq 2$ vertices (n even) and of degree $\Delta \geq 1$ is denoted by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2-1$ and $l = 1, \ldots, \Delta$, there is an edge with the label l between the vertex $(1, j)$ and $(2, (j + 2^{l-1} - 1) \bmod n/2)$.*

From the definition it follows that this graph is connected only if $\Delta \geq 2$. We will consider the Knödel graphs with $\Delta = \lceil \log_2 n \rceil$.

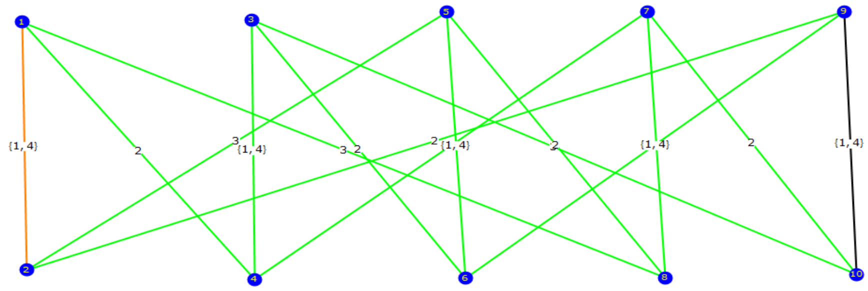

Fig. 4. Gossip graph based on Knödel graph

In [18] it was shown that the combination of two Knödel graphs with $\Delta_1 = \lceil \log_2 n \rceil$ and $\Delta_2 = 1$ is a complete gossip graph (see Fig. 4). Let us define the operation $A_l^- = \{P^-(e) | t_G(e) < t_G(l)\}$ as an operation of permutation of all edges that have lower weight value (or label) then l. After application of A_2^- on Knödel graphs all edges with label 1 will be permuted(see Fig. 5).

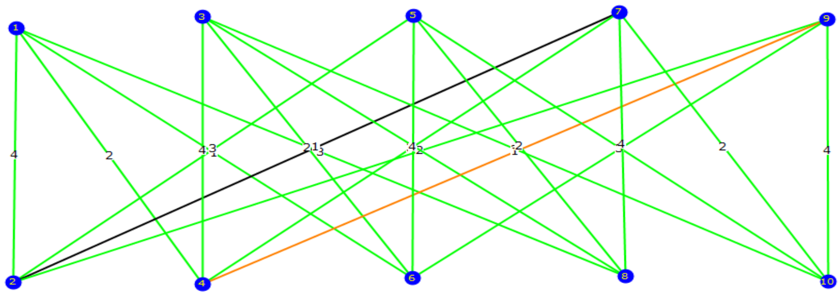

Fig. 5. NOHO graph based on Knödel graph

Lemma 3: *The result of application of A_2^- on complete gossip graph based on Knödel graph is a NOHO gossip graph with minimum gossiping time $T = \lceil \log_2 n \rceil$.*

Proof. Without loss of generality we can consider only the permutation of an edge that connects vertices $(1, 0)$ and $(2, 0)$. This edge has two adjacent edges with label 2, hence, after A_2^- it will change its position twice. So, after this operation the vertices that connect this edge are $(2, 1)$ and $(1, n/2 - 1)$. These vertices were not connected in our initial graph (according to the definition of Knödel graph), hence, after A_2^- there will be no duplicate edges. On the other hand, here we apply an inverse operation of the operation described in the previous section of case 2. Really, after permutation of this edge the vertices $(1, 0)$, $(2, 0)$, $(2, 1)$ and $(1, n/2 - 1)$ and edges with labels $t_G(e_1) = 1$, $t_G(e_2) = 2$, $t_G(e_3) = 2$ and $t_G(e_4) = \lceil \log_2 n \rceil$ will form a tetragon which is not a cycle. Thus, we exclude the possibility of existence of cycles after this operation. Hence, after application of A_2^- we obtain a NOHO graph which is equivalent to our initial complete gossip graph. ■

This methods of local interchange also have some other applications, which allowed us to investigate many variations of broadcast problem and improve some known parameters concerning the broadcast schemes, but it is not the subject of this paper.

Acknowledgements

This work was supported by State Committee of Science (SCS) MES RA, in frame of the research project No SCS 13-1B170. The authors are grateful to Akad. Yu.H. Shoukourian for important discussions and critical remarks on all stages of the work.

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Submitted 02.09.2013, accepted 18.10.2013.

Gossip խնդիրների հետազոտումը “Լոկալ փոխանակման“ մեթոդի միջոցով

Վ. Հովնանյան, Ս. Պողոսյան և Վ. Պողոսյան

Ամփոփում

Այս հոդվածում նկարագրված են *լոկալ փոխանակման* մեթոդը և նրա կիրառությունները gossip խնդիրների հետազոտման համար: Այս մեթոդը հիմնված է *տեղափոխել ավելի մեծերը* և *տեղափոխել ավելի փոքրերը* գործողությունների բազմակի կիրառման վրա, որոնք արտապատկերում են մի n գագաթանի gossip գրաֆը մեկ ուրիշի վրա՝ տեղափոխելով միայն կողերը, չփոխելով կողերի նիշերը (համապատասխան զանգերի ժամանակները): Օգտագործելով այս մեթոդը՝ **Knödel** գրաֆների հիման վրա մենք ստացել ենք մինիմալ ժամանակային NOHO gossip գրաֆներ (ոչ ոք երբեք չի լսում իր ինֆորմացիան): Ստացված մինիմալ ժամանակի ճշգրիտ արժեքն է՝ $T = \lceil \log n \rceil$: Այս մեթոդի միջոցով տրվել է gossip սխեմաների մինիմալ անհրաժեշտ զանգերի քանակի բանաձևի $(2n - 4, n \geq 4)$ ևս մի նոր արտածում:

Исследование Gossip проблем при помощи метода "Локальных перестановок"

В. Овнанян, С. Погосян и В. Погосян

Аннотация

В этой статье введен метод *локальных перестановок* для изучения gossip задач. Этот метод основан на многократном применении операций *переставить более старшие* и *переставить менее младшие*, которые отображают один граф с n вершинами на другой, перемещая только ребра, не меняя их меток (время соответствующего звонка). Используя этот метод, на основе **Knödel** графов, мы получили минимально временные NOHO gossip графы (никто никогда не услышит свою информацию). Точное значение полученного минимального времени $T = \lceil \log n \rceil$. С помощью этого метода получено еще одно новое доказательство формулы $(2n - 4, n \geq 4)$ для минимального необходимого количества звонков gossip схем.