Mathematical Problems of Computer Science 41, 103--113, 2014.

On an Algebraic Classification of Multidimensional Recursively Enumerable Sets Expressible in Formal Arithmetical Systems[1]

Seda N. Manukian

Institute for Informatics and Automation Problems of NAS RA
e-mail: zaslav@ipia.sci.am

Abstract

Algebraic representations of multidimensional recursively enumerable sets which are expressible in formal arithmetical systems based on the signatures $(0,=,S,+)$, $(0,=,<,S)$, $(0,=,S)$, where $S(x)=x+1$, are introduced and investigated. The equivalence is established between the algebraic and logical representations of multidimensional recursively enumerable sets expressible in the mentioned systems.

Keywords: Predicate formula, Universal algebra, Recursively enumerable set, Mathematical structure, Deductive system, Formal arithmetic.

1. Introduction

In this paper algebraic representations of multidimensional recursively enumerable sets (RES) described in some subsystems of Peano's formal arithmetic ([1], [2], [3]) are introduced and investigated. Similar problems concerning two-dimensional RESes are considered in [4], [5], [6]. But the structure of algebraic representations of multidimensional RESes differs from the structure of algebraic representations of two-dimensional ones. It was necessary to introduce essential changes in the notions used in [4], [5], [6] for the description of such algebraic representations. However, as it will be proved below, the relations between algebraic and logical representations of multidimensional RESes are similar to those described in [5]. Theorems 2.1, 2.2, 2.3 (see below) about such relations will be formulated in Sec.2 and proved in Sec.3.

2. Main Definitions and Results

Let us give the definitions of notions used below (cf. [7], [8]). An n-dimensional arithmetical set, where $n \geq 1$, is defined in a natural way as a set of n-tuples $(x_1, x_2,..., x_n)$, where $x_1, x_2,..., x_n$ are

[1] This work was supported by State Committee of Science, MES RA, in frame of the research project № SCS 13-B321.

nonnegative integers 0, 1, 2, ... An n-dimensional arithmetical predicate is defined as a predicate which is true on some n-dimensional arithmetical set and false out of it.

The notion of recursively enumerable set (RES) is defined as in [1]. An algebra is defined as a "universal algebra" ([9], [10]) with a fixed set of basic elements. Thus, any algebra is described by a main set M, by a set of operations $f_1, f_2,...$ on M (in general not everywhere defined), and a set of basic elements $a_1, a_2,...$ in M ([5]). We say that an element $a \in M$ is inductively representable in a given algebra $(M; f_1, f_2,...; a_1, a_2,...)$ if it can be obtained by the operations $f_1, f_2,...$ from the basic elements $a_1, a_2,...$. The notions of a subalgebra and a proper subalgebra of a given algebra are defined in a natural way (for example, as in [5]).

We will consider the following operations on multidimensional RESes (cf [14]).

1) The operations of union $\cup$ and intersection $\cap$ of RESes are defined in a usual way (note that these operations are applied only to RESes having equal dimensions).
2) The operation $\downarrow_i$ of projection for n-dimensional RES A concerning i-th co-ordinate, where $1 \le i \le n$, is defined by the following generating rule (g.r.): if $(x_1, x_2,..., x_n) \in A$, then $(x_1, x_2,..., x_{i-1}, x_{i+1},..., x_n) \in \downarrow_i (A)$.
3) The operation $\times$ of Cartesian product for n-dimensional RES A and m-dimensional RES B is defined by the following g.r.: if $(x_1, x_2,..., x_n) \in A$, and $(y_1, y_2,..., y_m) \in B$, then $(x_1, x_2,..., x_n, y_1, y_2,..., y_m) \in A \times B$.
4) The operation T_{ij} of transposition of i-th and j-th co-ordinates in n-dimensional RES A, where $1 \le i, j \le n$, is defined by the following g.r.: if $(x_1, x_2,..., x_n) \in A$, then $(x_1, x_2,..., x_{i-1}, x_j, x_{i+1},..., x_{j-1}, x_i, x_{j+1},... x_n) \in T_{ij}(A)$.
5) The operation $*$ of transitive closure for a RES A having an even dimension 2n is defined by the following generating rules: (a) if $(x_1, x_2,..., x_{2n}) \in A$, then $(x_1, x_2,..., x_{2n}) \in *A$; (b) if $(x_1, x_2,..., x_n, y_1, y_2,..., y_n) \in *A$ and $(y_1, y_2,..., y_n, z_1, z_2,..., z_n) \in *A$, then $(x_1, x_2,..., x_n, z_1, z_2,..., z_n) \in *A$.

The following RESes are used as basic elements for the considered algebras (cf. [7]): $Z_0 = \{x \mid x = 0\}$; $R = \{(x, y) \mid y = x + 1\}$; $Add = \{(x, y, z) \mid z = x + y\}$; $Q = \{(x, y) \mid x < y\}$; $J = \{(x, y) \mid x \ne y\}$.

Examples: $Q = *R$; $*(\downarrow_1 (Add)) = \downarrow_1 (Add)$.

Let us define the algebras Θ^0, Θ_1, Θ_2, Θ_3.

The main set for these algebras is the set of all multidimensional RESes having the dimensions $n \ge 1$. The list of operations for all these algebras is $(\cup, \cap, \downarrow, \times, T_{ij})$. The lists of basic elements are as follows (cf. [7]): (Z_0, R, Add) for Θ^0, (Z_0, R, Q) for Θ_1, (Z_0, R, J) for Θ_2, (Z_0, R) for Θ_3.

Note: The introduced algebras are different from the algebras denoted by Θ^0, Θ_1, Θ_2, Θ_3 in [5]. The algebras having these notations in [5] we will denote below by $\tilde{\Theta}_0$, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$. The relations between the algebras Θ^0, Θ_1, Θ_2, Θ_3 and $\tilde{\Theta}_0$, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$ will be considered in Sec.3.

The notion of a predicate formula based on the logical operations $\&, \vee, \supset, \neg, \forall, \exists$ (as other notions connected with it, for example, the notion of a term) is defined in a usual way ([1], [3], [11], see also [5]). A signature is defined in a usual way as any set of constants symbols, functional symbols, predicate symbols. We say that a formula F (respectively, a term t) belongs to a given signature Γ (or is a formula (respectively, a term) in the signature Γ) if all the constants symbols, functional symbols, predicate symbols contained in F (respectively, all the constants symbols and functional symbols contained in t) belong to Γ. We will consider the signatures $(0, S, +, =)$, $(0, =, <, S)$, $(0, S, =)$, where S is an one-dimensional functional symbol; these signatures will be denoted below respectively by N_H, N_L, N_S (cf. [7]). Note that similar notations are used in [7] as the notations of the corresponding mathematical structures (however, the structure corresponding to the signature $(0, S, +, =)$ is denoted in [7] (and in [3]) by N_A). The arithmetical interpretation of a predicate formula belonging to one of these signatures and containing no other free variables except $x_1, x_2, ..., x_n$ is defined in a natural way as an n-dimensional arithmetical predicate; the functional symbol S is interpreted as the function $S(x) = x + 1$, and other symbols in the mentioned signatures are interpreted in a natural way. The deductive systems of formal arithmetic in the signatures N_H, N_L, N_S are defined as in ([1], [3], [11]-[13]; see also [6]); we will denote these deductive arithmetical systems respectively by Ded_H, Ded_L, Ded_S (cf. [6]). For example, the system Ded_H is equivalent to M. Presburger's system described in [11]-[13]. We say that formulas F and G (respectively terms t and s) are equivalent in the framework of the corresponding deductive system if the formula $(F \supset G) \& (G \supset F)$ (respectively, the formula $t = s$) is deducible in this system. If the formulas F and G or the terms t and s are equivalent in Ded_H (respectively, Ded_L or Ded_S), we will say that they are Ded_H-equivalent (respectively, Ded_L-equivalent or Ded_S-equivalent).

All mentioned systems of formal arithmetic are complete ([3], [11]-[13]). We say that a set Γ of predicate formulas belonging to one of the mentioned signatures admits the elimination of quantifiers (in the framework of the corresponding deductive system) if for any predicate formula F belonging to Γ a formula G belonging to Γ can be constructed so that G does not contain quantifiers and is equivalent to F in the framework of the corresponding deductive system. The sets of all predicate formulas belonging to the signatures N_H, N_L, N_S admit the elimination of quantifiers in the framework of the corresponding deductive systems Ded_H, Ded_L, Ded_S ([3], [11]-[13]). By $S^n(t)$, where $n \geq 0$, and t is a term, we denote the term $S(S(...S(t)...))$, where the symbol S is repeated n times ($S^0(t)$ is t). By $\overline{n}$ we denote the term $S^n(0)$. We say that a k-dimensional arithmetical set A is represented (or representable) by a formula F belonging to one of the mentioned signatures and containing free variables $x_1, x_2, ..., x_k$, if the following condition holds: the arithmetical interpretation of the formula obtained by the substitution of the terms $\overline{n}_1, \overline{n}_2, ..., \overline{n}_k$ for the variables $x_1, x_2, ..., x_k$ in F is true if and only if $(n_1, n_2, ..., n_k) \in A$. We say that a k-dimensional arithmetical set A is represented (or representable) in Ded_H (respectively, Ded_L, Ded_S) by a formula F in N_H (respectively N_L, N_S) if it is represented by some formula F' equivalent to F in Ded_H (respectively, Ded_L, Ded_S). For example, the (n+1)-dimensional RES $\{(x_1, x_2, ..., x_n, y) \mid x_1 + x_2 + ... + x_n < y\}$ is represented in Ded_H by the formula $\exists z(x_1 + x_2 + ... + x_n + z + S(0) = y)$ in N_H.

A formula F in the signature N_S is said to be positive if it contains no other logical symbols except $\exists, \&, \vee, \neg$ and all the symbols $\neg$ of negation contained in it relate to elementary subformulas containing no more than one variable (cf. [5], [7]).

Theorem 2.1: *A multidimensional RES is inductively representable in the algebra Θ^0 (respectively Θ_1, Θ_2) if and only if it is represented in Ded_H (respectively, Ded_L, Ded_S) by a formula in N_H (respectively, N_L, N_S).*

Theorem 2.2: *A multidimensional RES is inductively representable in the algebra Θ_3 if and only if it is represented in Ded_S by a positive formula in N_S.*

Theorem 2.3: *Every next algebra in the sequence Θ^0, Θ_1, Θ_2, Θ_3 is a proper subalgebra of the preceding one.*

Theorems 2.1 and 2.2 are formulated (without proofs and in some other terms) in [7].

3. Proofs of Theorems

In this section the proofs of Theorems 2.1, 2.2, 2.3 will be given.

We will consider the following sets. By V we denote the set of all non-negative integers 0,1,2,..., by V^k we denote the set of all k-tuples $(x_1, x_2, ..., x_k)$ where $k \geq 1$, and all x_i are non-negative integers. By O we denote the 1-dimensional empty set, by O^k we denote the k-dimensional empty set. By E and Q_1 we denote the sets $E = \{(x, y) \mid x = y\}$ and $Q_1 = \{(x, y) \mid x \leq y\}$. Obviously, all these sets are represented in the following deductive systems: V in Ded_S by the formula $x = x$, V^k in Ded_S by the formula $(x_1 = x_1 \,\&\, x_2 = x_2 \,\&\, ... \,\&\, x_k = x_k)$, O in Ded_S by the formula $x = S(x)$, O^k in Ded_S by the formula $(x_1 = S(x_1) \,\&\, x_2 = S(x_2) \,\&\, ... \,\&\, x_k = S(x_k))$, E in Ded_S by the formula $x = y$, Q_1 in Ded_L by the formula $(x < y) \vee (x = y)$.

Lemma 3.1: *The sets V, V^k, O, O^k, E, Q_1, Q, J are inductively representable in the following algebras: the sets V, V^k, O, O^k, E in all algebras Θ^0, Θ_1, Θ_2, Θ_3, the sets Q_1, Q - in the algebras Θ^0 and Θ_1, the set J - in the algebras Θ^0, Θ_1, Θ_2.*

The proof is given by the following equalities:
$V = \downarrow_2 (R)$; $V^k = V \times V \times ... \times V$, where the symbol V is repeated k times; $O = \downarrow_1 (R \cap T_{12}(R))$; $O^k = O \times O \times ... \times O$, where the symbol O is repeated k times; $E = \downarrow_2 ((R \times V) \cap (V \times T_{12}(R)))$; $Q_1 = Q \cup E$; $Q_1 = \downarrow_1 (Add)$; $Q = \downarrow_2 ((Q_1 \times V) \cap (V \times R))$; $J = Q \cup T_{12}(Q)$.

Corollary: *Every next algebra in the sequence Θ^0, Θ_1, Θ_2, Θ_3 is a subalgebra of the preceding one.*

Lemma 3.2: *Any RES inductively representable in* Θ^0 *(respectively,* Θ_1*,* Θ_2*) can be represented in* Ded_H *(respectively,* Ded_L*,* Ded_S*) by a formula in* N_H *(respectively,* N_L*,* N_S*).*

Proof: The basic sets for Θ^0 are represented by the formulas $x = 0$, $y = S(x)$, $z = x + y$; similarly, the basic sets for Θ_1 are represented by the formulas $x = 0$, $y = S(x)$, $x < y$; for Θ_2 - by the formulas $x = 0$, $y = S(x)$, $\neg(x = y)$. If arithmetical sets A and B having equal dimensions are represented by formulas F and G, then the sets $A \cup B$ and $A \cap B$ are represented by the formulas $(F \vee G)$ and $(F \,\&\, G)$. If an n-dimensional arithmetical set A is represented by a formula F containing free variables $x_1, x_2, ..., x_n$, then the set $\downarrow_i (A)$, where $1 \le i \le n$, is represented by the formula $\exists x_i (F)$. If an n-dimensional arithmetical set A is represented by a formula F containing only free variables $x_1, x_2, ..., x_n$, and an m-dimensional arithmetical set B is represented by a formula G containing only free variables $y_1, y_2, ..., y_m$, then the set $A \times B$ is represented by the formula $F \,\&\, G'$, where the formula G' is obtained from G by the substitution of variables $x_{n+1}, x_{n+2}, ..., x_{n+m}$ for $y_1, y_2, ..., y_m$ in G. If an n-dimensional arithmetical set A is represented by a formula F containing free variables $x_1, x_2, ..., x_n$, then the formula $T_{ij}(A)$, where $1 \le i, j \le n$, is represented by a formula F' obtained from F by a corresponding replacement of free variables. This completes the proof.

Now we will give the proof of the statement opposite to the statement of Lemma 3.2.

In what follows any term in N_H having the form $(x + x + ... + x)$, where the variable x is repeated k times, will be shortly denoted by kx. The notation kx will denote the term 0 when $k = 0$; it will denote the term x when $k = 1$.

We will consider below the following sets.

(1) The set Z_k, where k is a constant, $k \ge 0$; it is a one-dimensional set containing only the number k.

(2) The set W_k, where k is a constant, $k \ge 0$; it is a one-dimensional set containing all the numbers x such that $x > k$.

(3) The set R_k, where k is a constant, $k \ge 1$; it is a two-dimensional set $\{(x, y) \mid y = S^k(x)\}$.

(4) The set $EAdd_k$, where k is a constant, $k \ge 0$; it is a two-dimensional set $\{(x, y) \mid y = kx\}$.

(5) The set $Linexp\ (k_1, k_2, ..., k_n, q)$, where $n \ge 1$, and $k_1, k_2, ..., k_n, q$ are constants, $k_1 \ge 0$, $k_2 \ge 0$, ..., $k_n \ge 0$, $q \ge 0$; it is an (n+1)-dimensional set $\{(x_1, x_2, ..., x_n, y) \mid k_1 x_1 + k_2 x_2 + ... + k_n x_n + q = y\}$.

(6) The set $Congr_k(x, y)$, where k is a constant, $k \ge 2$; it is a two-dimensional set $\{(x, y) \mid (x \equiv y)(\bmod k)\}$.

Clearly, all these sets are represented by formulas in the following deductive systems: Z_k is represented by the formula $(x = \overline{k})$ in *Ded*$_H$, *Ded*$_L$, *Ded*$_S$; W_k is represented by the formula $\exists z(x = S^{k+1}(z))$ in *Ded*$_H$, *Ded*$_L$, *Ded*$_S$; R_k is represented by the formula $y = S^k(x)$ in *Ded*$_H$,

Ded_L, Ded_S; $EAdd_k$ is represented by the formula $y = kx$ in Ded_H; $Linexp(k_1,k_2,...,k_n,q)$ is represented in Ded_H by the formula $k_1x_1 + k_2x_2 + ... + k_nx_n + \overline{q} = y$; $Congr_k(x, y)$, where $k \geq 2$ is represented in Ded_H by the formula $\exists z((x + kz = y) \vee (y + kz = x))$.

Lemma 3.3: *The sets* Z_k, *where* $k \geq 0$, *the sets* W_k, *where* $k \geq 0$, *the sets* R_k, *where* $k \geq 1$, *the sets* $EAdd_k$, *where* $k \geq 0$, *the sets* $Linexp(k_1,k_2,...,k_n,q)$, *where* $n \geq 1$, $k_i \geq 0$ *for* $1 \leq i \leq n$, $q \geq 0$, *the sets* $Congr_k(x, y)$, *where* $k \geq 2$ *are inductively representable in the following algebras:* Z_k, W_k *and* R_k *in* Θ^0, Θ_1, Θ_2, Θ_3; $EAdd_k$, $Linexp(k_1,k_2,...,k_n,q)$ *and* $Congr_k(x, y)$ *in* Θ^0.

The proof is given by the following equalities (note that the sets Z_0 and R are included as basic elements in all the algebras Θ^0, Θ_1, Θ_2, Θ_3):

$Z_{k+1} = \downarrow_1 ((Z_k \times V) \cap R)$; $W_0 = \downarrow_1 (R)$; $W_{k+1} = \downarrow_1 ((W_k \times V) \cap R)$; $R_1 = R$;

$R_{k+1} = \downarrow_2 ((R_k \times V) \cap (V \times R))$; $EAdd_0 = V \times Z_0$;

$EAdd_{k+1} = \downarrow_1 (\downarrow_1 ((EAdd_k \times V^2) \cap (V \times Add) \cap T_{23}(E \times V^2)))$;

$Linexp(k, q) = \downarrow_2 (\downarrow_2 ((EAdd_k \times V^2) \cap (V^2 \times Z_q \times V) \cap (V \times Add)))$;

$Linexp(k_1, k_2,...,k_n, l, q) = \downarrow_{n+2} (\downarrow_{n+2} (T_{n+1,n+3}(Linexp(k_1, k_2,..k_n, q) \times V^3) \cap (V^n \times EAdd_l \times V^2) \cap (V^{n+1} \times Add)))$;

$Congr_k = \downarrow_1 (\downarrow_1 ((EAdd_k \times V^2) \cap (V \times Add))) \cup T_{12}(\downarrow_1 (\downarrow_1 ((Eadd_k \times V^2) \cap (V \times Add))))$.

Lemma 3.4: *Any term in* N_H *is* Ded_H*-equivalent to a term having the form* $k_1x_1 + k_2x_2 + ... + k_nx_n + \overline{q}$, *where* q *is a nonnegative integer constant. Any formula in* N_H *is* Ded_H*-equivalent to a formula which can be obtained by* $\&$ *and* $\vee$ *from subformulas having the form* $t < s$ *or* $(t \equiv s)(\bmod k)$, *where* t *and* s *are terms, and* k *is an integer constant,* $k \geq 2$.

This Lemma is proved (in other terms) in [3], [4], [11] (cf. [6], Lemma 4.1).

Lemma 3.5: *Any RES represented in* Ded_H *by a formula in* N_H *is inductively representable in the algebra* Θ^0.

Proof: Let F be a formula in N_H. Let us denote by $x_1, x_2,..., x_n$ the list of all free variables contained in F. Using Lemma 3.4 we conclude that there exists a formula F' which is Ded_H-equivalent to F and can be obtained by $\&$ and $\vee$ from subformulas having the form $t < s$ or $(t \equiv s)(\bmod k)$, where t and s have the form described in Lemma 3.4. Without loss of generality we can suppose that the list of variables in all mentioned terms t and s coincides with the list $x_1, x_2,..., x_n$ (indeed, if some variable x_i is missing in a corresponding sum, then we can add to this sum the summand $0 \cdot x_i$; the order of summands in all considered sums can be unified using the operation T_{ij}). We see that the formula F is Ded_H-equivalent to some formula which can be obtained by $\&$ and $\vee$ from the formulas having the form $t < s$ or $(t \equiv s)(\bmod k)$ in which all the terms t and s have the form $k_1x_1 + k_2x_2 + ... + k_nx_n + \overline{q}$, where $k_i \geq 0$ for $1 \leq i \leq n$, and $q \geq 0$. n-dimensional sets represented by the formulas of such kind can be described in Θ^0 by the following expressions: the set represented by the formula having the

form $t < s$ - by the expression of the form $\downarrow_{n+1} (\downarrow_{n+2} ((Linexp(k_1^{'}, k_2^{'},..., k_n^{'}, q') \times V) \cap T_{n+1,n+2}(Linexp(k_1^{"}, k_2^{"},..., k_n^{"}, q'') \times V) \cap (V^n \times Q)))$;
the set represented by the formula having the form $(t \equiv s)(\mathrm{mod}\, k)$ - by the expression of the form $\downarrow_{n+1} (\downarrow_{n+2} ((Linexp(k_1^{'}, k_2^{'},..., k_n^{'}, q') \times V) \cap T_{n+1,n+2}(Linexp(k_1^{''}, k_2^{''},..., k_n^{''}, q'') \times V) \cap (V^n \times Congr_k)))$.
Thus, any n-dimensional set represented by the formula F can be described in Θ^0 applying the operations $\cap$ and $\cup$ to the expressions having the forms mentioned above.
This completes the proof.

Lemma 3.6: *Any term in N_L has the form $S^k(x)$ or $S^k(0)$, where x is a variable. Any formula in N_L is Ded_L-equivalent to a formula which can be obtained by $\&$ and $\vee$ from subformulas having the form $(t < s)$, where t and s are terms.*

This Lemma is proved (in other terms) in [3], [5], [11] (cf. [6], Lemma 4.2).

Lemma 3.7: *Any RES represented in Ded_L by some formula in N_L is inductively representable in the algebra Θ_1.*

Proof: Let F be a formula in N_L. Let us denote by $x_1, x_2,..., x_n$ the list of all free variables in F. We suppose that $n > 2$ (the case $n \le 2$ is considered in a similar way). Using Lemma 3.6 we conclude that F is Ded_L-equivalent to some formula F' which can be obtained by $\&$ and $\vee$ from subformulas of the form $(t < s)$, where t and s are terms. Let us consider the case when t and s contain only the variables x_1 and x_2 (the general case is reduced to the mentioned one using the operation T_{ij}). We will denote the variables x_1 and x_2 by x and y. Using Lemma 3.6 we see that in the subformula $(t < s)$ the term t has one of the forms $S^k(x)$, $S^k(y)$, $S^k(0)$, where $k \ge 0$; the term s has one of the forms $S^l(x)$, $S^l(y)$, $S^l(0)$, where $l \ge 0$. Thus, there are 9 possible forms of the subformula $(t < s)$. If $(t < s)$ has the form $S^k(x) < S^l(y)$, then the n-dimensional set represented by this formula is $\downarrow_1 ((R_{k-l} \times V) \cap (V \times Q)) \times V^{n-2}$ when $k > l$; it is $\downarrow_3 ((Q \times V) \cap T_{23}(V \times R_{l-k})) \times V^{n-2}$, when $k < l$, and $Q \times V^{n-2}$ when $k = l$. If $(t < s)$ has the form $S^k(x) < S^l(x)$, then the n-dimensional set represented by this formula is O^n when $k \ge l$, and is V^n when $k < l$. If $(t < s)$ has the form $S^k(x) < S^l(0)$, then the n-dimensional set represented by this formula is O^n when $k \ge l$, and is $(Z_0 \cup Z_1 \cup ... \cup Z_{l-k-1}) \times V^{n-1}$ when $k < l$. The remaining forms of the formula $(t < s)$ are considered in a similar way. Thus, the n-dimensional RES represented by the formula F in Ded_L is obtained by $\cup$ and $\cap$ from sets inductively representable in Θ_1.
This completes the proof.

Lemma 3.8: *Any term in N_S has the form $S^k(x)$ or $S^k(0)$, where x is a variable. Any formula in N_S is Ded_S-equivalent to a formula which can be obtained by $\&$ and $\vee$ from subformulas having the form $(t = s)$ or $\neg(t = s)$, where t and s are terms.*

This Lemma is actually proved (in other terms) in [3], [11] (cf. [5], Lemma 3.8).

Lemma 3.9: *Any RES represented in Ded_S by formula in N_S is inductively representable in the algebra Θ_2.*

Proof: The proof is similar to that of Lemma 3.7. Let F be a formula in N_S. Let us denote by $x_1, x_2, ..., x_n$ the list of all free variables contained in F. We suppose (as in the proof of Lemma 3.7) that $n > 2$. Using Lemma 3.8 we conclude that F is Ded_S-equivalent to some formula F' which can be obtained by $\&$ and $\vee$ from subformulas having the forms $(t = s)$ and $\neg(t = s)$. As in the proof of Lemma 3.7 we consider the case when t and s contain only variables x_1 and x_2; we will denote these variables by x and y. Using Lemma 3.8 we see that in the subformulas $(t = s)$ and $\neg(t = s)$ the term t has one of the forms $S^k(x)$, $S^k(y)$, $S^k(0)$, where $k \geq 0$; the term s has one of the forms $S^l(x)$, $S^l(y)$, $S^l(0)$, where $l \geq 0$. Thus, there are 9 possible forms of the subformula $(t = s)$ and 9 possible forms of the subformula $\neg(t = s)$.
If $(t = s)$ has the form $S^k(x) = S^l(y)$, then the n-dimensional set represented by this formula is $R_{k-l} \times V^{n-2}$ when $k > l$; it is $T_{12}(R_{l-k}) \times V^{n-2}$ when $k < l$, and $E \times V^{n-2}$ when $k = l$.
The n-dimensional set represented by the formula $\neg(S^k(x) = S^l(y))$ is $\downarrow_2 ((R_{k-l} \times V) \cap (V \times J))$ when $k > l$, it is $T_{12}(\downarrow_2 ((R_{l-k} \times V) \cap (V \times J)))$ when $k < l$, and $J \times V^{n-2}$ when $k = l$.
The n-dimensional set represented by the formula $S^k(x) = S^l(x)$ is O^n when $k \neq l$; it is V^n when $k = l$.
The n-dimensional set represented by the formula $\neg(S^k(x) = S^l(x))$ is V^n when $k \neq l$; it is O^n when $k = l$.
The n-dimensional set represented by the formula $S^k(x) = S^l(0)$ is O^n when $k > l$; it is $Z_{l-k} \times V^{n-1}$ when $k \leq l$.
The n-dimensional set represented by the formula $\neg(S^k(x) = S^l(0))$ is V^n when $k > l$; it is $(Z_0 \cup Z_1 \cup ... \cup Z_{l-k-1} \cup W_{l-k}) \times V^{n-1}$ when $k < l$; and $W_0 \times V^{n-1}$ when $k = l$. The remaining forms of the formulas $(t = s)$ and $\neg(t = s)$ are considered in a similar way.
Thus, the n-dimensional RES represented by the formula F is obtained by $\cup$ and $\cap$ from sets inductively representable in Θ_2.
This completes the proof.

The proof of Theorem 2.1 is obtained now using Lemmas 3.2, 3.5, 3.7, 3.9.

Lemma 3.10: *The set of positive formulas in N_S admits the elimination of quantifiers in the framework of Ded_S.*

The proof follows from the considerations in [3], because it is easily seen that the method of elimination of quantifiers in N_S described in [3] gives for any positive formula F in N_S some positive formula G such that G does not contain quantifiers and is Ded_S-equivalent to F.

Lemma 3.11: *Any positive formula in N_S is Ded_S-equivalent to a formula which can be obtained by $\&$ and $\vee$ from subformulas having the form $(t = s)$ or $\neg(t = s)$, where t and s*

are terms of the form $S^k(x)$ *or* $S^k(0)$*, and any subformula of the form* $\neg(t=s)$ *contains no more than one variable.*

The proof is easily obtained using Lemmas 3.8 and 3.10.

Lemma 3.12: *Any RES inductively representable in* Θ_3 *can be represented in* Ded_S *by a positive formula in* N_S.

Proof: The basic sets in Θ_3 are represented in N_S by the positive formulas $x=0$ and $y=S(x)$. It is easily seen that the transformation of formulas generated in the proof of Lemma 3.2 by the operations $\cup, \cap, \times, \downarrow_i, T_{ij}$ gives positive formula being applied to positive formulas.
This completes the proof.

Lemma 3.13: *Any RES represented in* Ded_S *by a positive formula in* N_S *is inductively representable in the algebra* Θ_3.

Proof: Let F be a positive formula in N_S. Let us denote by $x_1, x_2, ..., x_n$ the list of all free variables in F. Similarly to the proof of Lemmas 3.7 and 3.9 we suppose that $n>1$. Using Lemma 3.11 we conclude that F is Ded_S-equivalent to a positive formula which can be obtained by & and $\vee$ from positive subformulas of the form $(t=s)$ or $\neg(t=s)$, where t and s are terms in N_S. It is easily seen that any n-dimensional set represented by a formula of the form $(t=s)$ is described by the expressions considered in the proof of Lemma 3.9. Now let us consider the sets represented by subformulas of the form $\neg(t=s)$. Let us recall that any positive formula of the form $\neg(t=s)$ contains no more than one variable. The single variable contained in $\neg(t=s)$ we denote by x and suppose that it coincides with the variable x_1 in the list $x_1, x_2, ..., x_n$ (the general case is considered similarly). Then the formula $\neg(t=s)$ has the form $\neg(S^k(x)=S^l(0))$, where $k \geq 0$, $l>0$. But the inductive representations of the set represented by this formula in Θ_2 are described in the proof of Lemma 3.9; it is easily seen that these representations are also representations in Θ_3. Thus, the n-dimensional RES represented by the formula F is obtained by $\cup$ and $\cap$ from sets inductively representable in Θ_3.
This completes the proof.

The proof of Theorem 2.2 is obtained now using Lemmas 3.12 and 3.13.

Lemma 3.14: *Any multidimensional RES is inductively representable in* $\tilde{\Theta}_0$ *(respectively,* $\tilde{\Theta}_1$, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$*) if and only if it is two-dimensional and is inductively representable in* Θ_0 *(respectively,* Θ_1, Θ_2, Θ_3*).*

Proof: Let us recall that we denote by $\tilde{\Theta}_0$, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$ the algebras denoted in [5] by Θ_0, Θ_1, Θ_2, Θ_3. As it is proved in [5], (in other terms) a two-dimensional RES is inductively representable in $\tilde{\Theta}_0$ (respectively, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$) if and only if it is represented in Ded_H

(respectively, Ded_L, Ded_S) by a formula in N_H (respectively, N_L, N_S); a two-dimensional RES is inductively representable in $\tilde{\Theta}_3$ if and only if it is represented in Ded_S by a positive formula in N_S. Now the statement of Lemma 3.14 is obtained using Theorems 2.1 and 2.2 proved above.

Proof of Theorem 2.3: As it is proved in [5], there exists a two-dimensional RES A_1 (respectively, A_2, A_3) which is inductively representable in $\tilde{\Theta}_1$ (respectively, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$) but not in $\tilde{\Theta}_0$ (respectively, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$). The statement of Theorem 2.3 is now obtained using Lemma 3.14.

Let us note that the list of operations in the algebras Θ_0, Θ_1, Θ_2, Θ_3 is similar to that considered in [14]. Let us note that if we add the operation of transitive closure to this list then any multidimensional RES will be inductively representable in the extended algebra Θ_3. Such statement is proved in [15] (see [15], Lemma 1).

Acknowledgement

The author is grateful to Professor Patrick Cegielski for setting the problem considered in this paper, for his attention to this work, for valuable advice and notes.

References

[1] S. C. Kleene, *Introduction to Metamathematics*, D. Van Nostrand Comp., Inc., New York-Toronto, 1952.

[2] E. Mendelson, *Introduction to Mathematical Logic*, D. Van Nostrand Comp., Inc., Princeton-Toronto-New York-London, 1964.

[3] H. Enderton, *A Mathematical Introduction to Logic*, 2nd ed., San Diego, Harcourt, Academic Press, 2001.

[4] S. N. Manukian, “Algebras of recursively enumerable sets and their applications to fuzzy logic”, *Journal of Mathematical Sciences*, vol. 130, no. 2, pp. 4598-4606, 2005.

[5] S. N. Manukian. “On the representation of recursively enumerable sets in weak arithmetics”, *Transactions of the IIAP of NAS RA, Mathematical Probelms of Computer Science*, vol. 27, pp. 90--110, 2006.

[6] S. N. Manukian, “Some algebraical and logical properties of two-dimensional arithmetical sets representable in Presburger’s System”, *Transactions of the IIAP of NAS RA, Mathematical Probelms of Computer Science*, vol. 37, pp. 64--74, 2012.

[7] S. Manukian, “On the inductive representation of many-dimensional recursively enumerable sets definable in some arithmetical structures”, *Proceedings of the International Conference “Computer Science and Information Technologies”, CSIT-09*, Yerevan, Armenia, pp. 51--53, 2009.

[8] S. N. Manukian. “Classification of many-dimensional arithmetical sets represented in M. Presburger’s system”, *Reports of the National Academy of Science of Armenia*, (in Russian), vol. 111, no. 2, pp. 114--120, 2011.

[9] G. Graetzer, *Universal Algebra*, 2nd Edition, New-York-Heidelberg-Berlin, 1979.

[10] A. I. Malcev, *Algebraic Systems*, Springer Verlag, 1973.

[11] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Band I.Zweite Auflage, Berlin-Heidelberg-New York, Springer Verlag, 1968.

[12] M. Presburger, “Über die Vollständigkeit eines gewissen System der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt”, *Comptes Rendu de I Congres der Mathematiciens des Pays Slaves*, Warszawa, pp. 92--101, 1930.
[13] R. Stansifer, *Presburger’s Article on Integer Arithmetic: Remarks and Translation*, Department of Computer Science, Cornell University, Ithaca, New York, 1984.
[14] G. S. Tseytin, “One method of representation for the theory of algorithms and enumerable sets”, *Transactions of Steklov Institute of the Russian Academy of Sciences*, (in Russian), vol. 72, pp. 69--98, 1964.
[15] S. N. Manukian. “On the structure of recursively enumerable fuzzy sets”, *Transactions of the IIAP of NAS RA and YSU, Mathematical Probelms of Computer Science*, (in Russian), vol. 17, pp. 86--91, 1997.

Submitted 14.12.2013, accepted 24. 02. 2014.

Ձևայնացված թվաբանական համակարգերում արտահայտելի բազմաչափ անդրադարձ թվարկելի բազմությունների որոշ հանրահաշվական դասակարգման մասին

Ս. Մանուկյան

Ամփոփում

Սահմանվում և հետազոտվում են $(0,=,S,+)$, $(0,=,<,S)$, $(0,=,S)$ (որտեղ $S(x)=x+1$) սիգնատուրաների վրա հիմնված ձևայնացված թվաբանության համակարգերի մեջ արտահայտելի բազմաչափ անդրադարձ թվարկելի բազմությունների հանրահաշվական ներկայացումները: Հաստատվում է համարժեքություն նշված տիպի անդրադարձ թվարկելի բազմությունների հանրահաշվական և տրամաբանական ներկայացումների միջև:

Об алгебраической классификации многомерных рекурсивно перечислимых множеств, выразимых в формальных арифметических системах

С. Манукян

Аннотация

Вводятся и исследуются алгебраические представления многомерных рекурсивно перечислимых множеств, которые выразимы в системах формальной арифметики, основанных на сигнатурах $(0,=,S,+)$, $(0,=,<,S)$, $(0,=,S)$, где $S(x)=x+1$. Устанавливается эквивалентность между алгебраическими и логическими представлениями многомерных рекурсивно перечислимых множеств, выразимых в указанных системах.