Mathematical Problems of Computer Science 45, 106--110, 2016.

Construction of Double ± 1 Error Correcting Linear Optimal Codes over Rings Z_7 and Z_9

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Abstract

In this paper a construction of double ± 1 error correcting linear optimal codes over rings Z_7 and Z_9 is presented.

Keywords: Error correcting codes, Codes over the rings Z_7 and Z_9, Asymmetrical errors.

1. Introduction

From practical point of view the codes over rings Z_{2m} or Z_{2m+1} are interesting, because they can be used in 2^{2m} – QAM (Quadrature amplitude modulation) schemes. Codes over finite rings, particularly over integer residue rings and their applications in coding theory have been studied for a long time. Errors happening in the channel are basically asymmetrical; they also have a limited magnitude and this effect is particularly applicable to flash memories.

There have been a couple of papers regarding to optimal ± 1 single error correcting codes over alphabet Z_m [1,2]. Also there are many linear codes capable to correct up to two errors of type ± 1 for different alphabets which have been found by computer search, but they are not optimal. The optimality criteria for the linear codes over fixed ring Z_m can be considered in two ways (see in [3]). First of all, recall that the code of the length n is optimal-1 if it has a minimum possible number of parity check symbols. Secondly, optimality-2 criteria for the code is that for a given number of parity check symbols, it has a maximum possible length. The linear code (12, 8) correcting double errors over ring Z_5 of value ± 1 presented in [3] satisfies the optimality criteria -1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 3 & 2 & 4 & 4 & 2 & 3 & 2 & 4 & 4 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 2 & 4 & 4 & 2 & 0 & 4 \end{bmatrix},$$

this code was given by the parity check matrix H, which has 8 information and 4 parity check symbols.

At this point we do not know any codes that satisfy the optimality criteria-2. In [3] a method how to compare two code constructions over different size of alphabets when both satisfy the optimality –1 criteria has been presented. Two factors are considered, namely:

1) The first factor should be the rate of the code, i.e., the ratio of the number of information symbols over the length of the code.
2) Second, the ratio between the numbers of possible amplitude errors corrected by the code over the size of alphabet minus 1, which corresponds to the number of all possible amplitude errors.

The product of these two factors is chosen as a merit to compare optimal codes over different size of alphabets [3].

For the code over ring Z_5 mentioned above the product is: $(8/12) * (2/4) = 0.3333$.

In this paper a constructions of the optimal-1 codes (16, 12) and (20, 16) over the rings Z_7 and Z_9 correcting double ± 1 errors is presented. For these codes the products will be (12/16) * (2/6) = 0.25 for code over Z_7 and (16/20) * (2/8) = 0.2 for Z_9 . These products are a little bit smaller than those for the code (12, 8) over ring Z_5, although there are much better ones compared to the codes over Z_{16} and Z_{128} in [2, 4, 5].

2. **Construction of Optimal (16, 12) Linear Code over Ring Z_7**

Our purpose is to construct an optimal linear code over ring Z_7 correcting double errors of the type ± 1. It is well known, that a linear code given by the parity check matrix H, can correct up to two errors of the type ± 1, only when H has a property according to which all the syndromes resulting from adding and subtracting operations between any two columns of the matrix H are different $\left(\pm h_i \pm h_j, where \; \left(h_i \neq \pm h_j\right)\right)$. For constructing this kind of matrix H, at first we will find a difference set in Z_7. For example, a difference set for a linear code (12, 8) constructed in [3] is the set {3,2,4,4,2}. A difference set is defined to have a property that the differences for any 2 components in the set are different in Z_5 given that difference is taken for the elements located at the same distance from each other where the distance itself can be from the set (1, 2, 3, 4). Note also that the distance between positions of elements is calculated modulo 5 in this case. For an example if the distance is chosen to be three, we have to take a difference between the 4-th and 1st positions of the set which is equal to 1, a difference between the 5-th and 2 nd positions of the set will be 0, a difference between the 1 st(6-th) and 3 rd positions of the set will be -1(4), a difference between the 2 nd(7-th) and 4-th positions of the set will be -2(3), and finally a difference between the 3 rd(8-th) and 5-th positions of the set will be 2.

For the ring Z_7 it is easy to check that the difference set is a set $-\{4,3,6,6,3,4,2\}$ of the length 7. For instance, for the distance equal to 1 all the corresponding differences resulting from$\left(3-4=\mathbf{6},\ 6-3=\mathbf{3},\ 6-6=\mathbf{0},\ 3-6=\mathbf{4},\ 4-3=\mathbf{1},\ 2-4=\mathbf{5},\ 4-2=\mathbf{2}(\boldsymbol{mod 7})\right)$ are different. 4-corresponds to the position with index 0 and the last position 2 corresponds to the position with index 6 and $0-6\ =1(\boldsymbol{mod 7})$ (all operations are in Z_7).

A linear (16, 12) code over ring Z_7 is given by the following parity check matrix H:

$$H=\begin{bmatrix}1&1&1&1&1&1&1&0&1&2&3&4&5&6&1&1\\6&5&4&3&2&1&0&2&2&2&2&2&2&2&1&1\\4&3&6&6&3&4&2&4&3&6&6&3&4&2&1&6\\1&1&1&1&1&1&1&4&3&6&6&3&4&2&0&0\end{bmatrix}. \qquad (1)$$

An approach how this matrix is designed is similar to one in [3]. It consists of three parts, namely the first 7 columns, the next 7 columns and a tail of two last columns. The first two rows of the first and second parts is a code correcting one error of the type ± 1, the third rows of the parts 1 and 2 as well as the forth row of the second part are a difference set for Z_7. It can be checked that a linear code over Z_7, given by the parity check matrix H in (1) can correct up to two errors of the type ± 1. This can be done in the similar manner demonstrated in [3] and, of course, also by computer.

Lemma 2.1: *A linear code (16, 12) correcting up to two errors of the type ± 1 is optimal in the sense that it has a minimal possible number of parity check symbols.*

Proof: In this case the number of combinations for each code word that can be corrected is

$$(1\ +\ 16*2\ +\ (16\, choose\ 2)*4)\ =513.$$

Thus, we have that $513*7^{12}\le 7^{16}$ and the cardinality of the best possible code is

$$7^{16}/513<7^{13}\ .$$

3. Construction of Optimal (20, 16) Linear Code over Ring Z_9

In this section we will construct an optimal (20, 16) linear code over ring $\boldsymbol{Z_9}$. As in previous construction we need to find a difference set of length 9 for $\boldsymbol{Z_9}$. In this case we could not find a difference set of length 9. So, to fix this problem we find a difference set of the length 8: $\{7,3,2,4,4,2\ ,3,7\}$. Similarly, in this set, for all distances (1, 2, 3, 4…) the differences of any 2 components should be different in Z_9. For instance, for the distance - 1 all the corresponding differences resulting from$\big(3-7=\mathbf{5},\ 2-3\ =\mathbf{8},\ 4-2\ =\mathbf{2},\ 4-4=\mathbf{0},\ 2-4\ =\mathbf{7},\ 3-2\ =\mathbf{1},\ 7-3\ =\mathbf{4}\ \ (\boldsymbol{mod 9})\big)$ are different (all operations are in Z_9).

In the previous construction, sequences consisting of all integers in Z_7- $\{0,1,2,3,4,5\ ,6\}$ have been used in rows of the matrix. Since we have for Z_9 a difference set with only 8 components, we should take either a sequence $\{0,1,2,3,4,5\ ,6,7\}$ *or* $\{\ 1,2,3,4,5\ ,6,7,8\ \}$.

In this case the parity check matrix for an optimal linear code (20, 16) correcting double errors of the type ± 1 has the following form:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 2 & 4 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 4 \\ 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 6 & 3 & 7 & 2 \end{bmatrix}. \quad (2)$$

As for previously constructed codes a corresponding parity check matrix (2) also consists of three parts. Since a difference set in this case has only 8 elements a corresponding split between those parts will be (8, 8, 4). This is because our target is to have a code of the length 20 which will be optimal and, therefore, we will need for the tail part to have 4 columns which are, in fact, the last four columns of the matrix (2). A linear code over Z_9, given by the parity check matrix H (2), can correct up to two errors of the type ± 1. The proof of this statement can be made in a similar manner demonstrated in [3], as well as by computer.

Lemma 3.1: *A linear code (20, 16) given by* (2) *correcting up to two errors of the type* ± 1 *is optimal in the sense that it has a minimal possible number of parity check symbols.*

Proof: In this case the number of combinations for each code word that can be corrected is

$$(1 + 20 * 2 + (20\, choose\, 2) * 4) = 801\,.$$

Thus, we have that $801 * 9^{16} \leq 9^{20}$ and the cardinality of the best possible code is $9^{20}/801 < 9^{17}$.

4. Conclusion

In this paper a construction of optimal double ± 1 error correcting linear optimal codes over rings Z_7 and Z_9 are constructed. We plan to investigate if an approach presented in this paper can be extended for the construction of codes for larger alphabets as well as for the construction of near optimal codes with higher code rates.

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Submitted 17.11.2015, accepted 10.02.2016

Z_7 և Z_9 օղակներում ± 1 մեծությամբ կրկնակի սխալ ուղղող օպտիմալ կոդերի կառուցում

Գ. Խաչատրյան և Հ. Խաչատրյան

Ամփոփում

Այս հոդվածի շրջանակներում ներկայացված են Z_7 և Z_9 օղակներում ± 1 մեծությամբ կրկնակի սխալ ուղղող օպտիմալ կոդերի կառուցումներ:

Построение оптимальных кодов в кольцах Z_7 и Z_9 исправляющие двойные ошибки размера ± 1

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Аннотация

В данной статье представлено построение оптимальных кодов в кольцах Z_7 и Z_9 исправляющие двойные ошибки размера ± 1.