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On Cyclability of Digraphs with a Manoussakis-type Condition

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Abstract

Let D be a digraph of order $n \geq 4$ and Y be a non-empty subset of vertices of D. Let for any pair u, v of distinct vertices of Y the digraph D contain a path from u to v and a path from v to u. Suppose D satisfies the following conditions for every triple $x, y, z \in Y$ such that x and y are nonadjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^+(z) + d^-(x) \geq 3n - 2$. We prove that there is a directed cycle in D which contains all the vertices of Y, except possibly one. This result is best possible in some situations and gives an answer to a question of Li, Flandrin and Shu (Discrete Mathematics, 307 (2007) 1291-1297).

Keywords: Digraphs, Cycles, Hamiltonian cycles, Cyclability.

1. Introduction

For convenience of the reader, terminology and notations will be given in details in section 2. A set S of vertices in a directed graph D (an undirected graph G) is said to be cyclable in D (in G) if D (if G) contains a directed cycle (undirected cycle) through all the vertices of S. There are many well-known conditions which guarantee the cyclability of a set of vertices in an undirected graph. Most of them can be seen as restrictions of Hamiltonian conditions to the considered set of vertices (See [1, 2, 3, 4]). Let's provide some examples below:

Theorem A: (R. Shi [3]). *Let G be a 2-connected undirected graph of order n. If S is a subset of the vertices of G and $d(x) \geq n/2$ for all vertices $x \in S$, then S is cyclable in G.*

Theorem B: (R. Shi [3]). *Let G be a 2-connected undirected graph of order n. If S is a subset of the vertices of G and $d(x) + d(y) \geq n$ for any two nonadjacent vertices $x \in S$ and $y \in S$, then S is cyclable in G.*

Notice that Theorems A and B generalize the classical theorems on hamiltonicity of Dirac and Ore, respectively. In view of the next theorems we need the following definitions.

Let D be a digraph of order $n \geq 3$ and S be a non-empty subset of vertices of D. Following [5], we say that a digraph D is S-strongly connected if for any pair x, y of distinct vertices of S the digraph D contains a path from x to y and a path from y to x.

A Meyniel set M is a subset of vertices of a digraph D such that $d(x) + d(y) \geq 2n - 1$ for every pair of distinct vertices x, y in M which are nonadjacent in D.

For general directed graphs (digraphs) there are not as many conditions in literature as for undirected graphs that guarantee the existence of a directed cycle with the given properties (in particular, sufficient conditions for the existence of a Hamiltonian cycles in digraphs). The more general and classical ones are the following theorem of M. Meyniel:
Theorem C: (M. Meyniel [6]). *Let D be a strongly connected digraph of order $n \geq 2$. If the vertex set of D is a Meyniel set, then D is Hamiltonian.*

Notice that Meyniel's theorem is a generalization of well-known classical theorems of Ghouila-Houri [7] and Woodall [8]. A beautiful short proof of Meyniel's theorem can be found in [9] (see also [10], pp. 399-400).

In [11], the following was proved:
Theorem D: (S. Darbinyan [11]). *Let D be a strongly connected digraph of order $n \geq 3$ and Y be a subset of vertices of D. If $|Y| = n-1$ and Y is a Meyniel set, then D is Hamiltonian or contains a cycle of length $n-1$.*

From Theorem D we obtain the following corollaries.
Corollary 1: *Let D be a strongly connected digraph of order $n \geq 3$. If D has $n-1$ vertices of degree at least n, then D is Hamiltonian or contains a cycle of length $n-1$.*
Corollary 2: *Let D be a strongly connected digraph of order $n \geq 3$ and Y be a subset of vertices of D. If $|Y| = n-1$ and Y is a Meyniel set, then D has a cycle that contains all the vertices of Y.*

A sufficient condition for cyclability in digraphs with the condition of Meyniel's theorem was given by K. A. Berman and X. Liu [12]. They improved Theorem F by proving the following generalization of the well-known theorem of Meyniel.
Theorem E: (K. Berman and X. Liu [12]). *Let D be a strongly connected digraph of order n. Then every Meyniel set M of D lies in a directed cycle.*

Later H. Li, E. Flandrin and J. Shu [5] proved the following generalization of Theorem E.
Theorem F: (H. Li, E. Flandrin and J. Shu [5]). *Let D be a digraph of order n and M be a Meyniel set in D. If D is M-strongly connected, then D contains a cycle through all the vertices of M.*

Let D be a digraph of order n. We say that a non-empty subset Y of the vertices of D satisfies condition A_0 if for every triple of the vertices x, y, z in Y such that x and y are nonadjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$.

Y. Manoussakis [13] proved a sufficient condition for hamiltonicity of digraphs that involves triples rather than pairs of vertices.
Theorem G: (Y. Manoussakis [13]). *Let D be a strongly connected digraph of order $n \geq 4$. If $V(D)$ satisfies condition A_0, then D is Hamiltonian.*

H. Li, Flandrin and Shu [5] put a question to know if this theorem of Manoussakis (or the sufficient conditions of hamiltonicity of digraphs of Bang-Jensen, Gutin and Li [14] or of Bang-Jensen, Guo and Yeo [15]) has a cyclable version.

In this paper we prove the following theorem which gives some answers to the above question when a subset $Y \neq \emptyset$ of the vertices of a digraph D satisfies condition A_0 and the digraph D is Y-strongly connected.
Theorem 1: *Let D be a digraph of order $n \geq 4$ and let Y be a non-empty subset of the vertices of D. Suppose that D is Y-strongly connected and the subset Y satisfies condition A_0. Then D contains a cycle through all the vertices of Y maybe except one.*
Remark 1: *The following example shows that there is a digraph D which contains a*

nonempty subset Y of $V(D)$ such that D is Y-strongly connected and the subset Y satisfies condition A_0 but D has no cycle that contains all the vertices of Y.

To see this, let G and H be two arbitrary disjoint digraphs with $|V(G)| = m \geq 2$ and $|V(H)| = n - m \geq 4$. Let $y \in V(H)$ and $x, z \in V(G)$, $x \neq z$. Assume that $d(y, H) = 2(n - m - 1)$, G contains a Hamiltonian cycle, $d^+(x, G) = m - 1$ and $d(z, G) = 2(m - 1)$. From G and H we form a new digraph D with $V(D) = V(G) \cup V(H)$ as follows: add all the possible arcs ux, xu, where $u \in V(H) \setminus \{y\}$, and the arc yx. An easy computation shows that

$$d(y) + d(z) + d^-(y) + d^+(x) = 4n - m - 6 \geq 3n - 2,$$

since $m \leq n - 4$. Thus, we have that the set $Y = \{x, y, z\}$ satisfies condition A_0. D is Y-strongly connected and has no cycle that contains all the vertices of Y.

Our proofs are based on the arguments of [5, 13].

2. Terminology and Notation

In this paper we consider finite digraphs without loops and multiple arcs. Terminology and notations not described below follow [16]. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The order of D is the number of its vertices. For any $x, y \in V(D)$, we also write $x \to y$ if $xy \in A(D)$. If $xy \in A(D)$, then we say that x dominates y or y is an out-neighbour of x and x is an in-neighbour of y. If $x \to y$ and $y \to z$ we write $x \to y \to z$. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). If there is no arc from x to y we shall use the notation $xy \notin A(D)$.

We let $N^+(x)$, $N^-(x)$ denote the set of out-neighbours, respectively the set of in-neighbours of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$ and $N^-(x, A) = A \cap N^-(x)$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. If $x \in V(D)$ and $A = \{x\}$ we sometimes write x instead of $\{x\}$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A \rangle$ or $\langle A \rangle$ for brevity.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). The length of a cycle or path is the number of its arcs. We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. An (x, y)-path P is an (X, Y)-path if $x \in X$, $y \in Y$ and $V(P) \cap (X \cup Y) = \{x, y\}$, where X and Y are some subsets of the vertices of a digraph D.

Given a vertex x of a directed path P or a directed cycle C, we use the notations x^+ and x^- for the successor and the predecessor of x (on P or on C) according to the orientation and in case of ambiguity, we precise P or C as a subscript (that is x_P^+ ...).

A cycle (respectively, a path) that contains all the vertices of D is a Hamiltonian cycle (respectively, is a Hamiltonian path). A digraph is Hamiltonian if it contains a Hamiltonian cycle. For a cycle $C := x_1 x_2 \cdots x_k x_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. If C is a cycle and P is a path in a digraph D, often we will write C instead of $V(C)$ and P instead of $V(P)$. A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y.

Let C be a non-Hamiltonian cycle in a digraph D. For the cycle C, a C-bypass is a path of length at least two with both end-vertices on C and no other vertices on C. If (x, y)-path P is a C-bypass with $V(P) \cap V(C) = \{x, y\}$, then we call the length of the path $C[x, y]$ the gap of P with respect to C.

If we consider a subset of vertices $S \subseteq V(D)$, we denote the vertices of S by S-vertices and the number of S-vertices in a cycle is called its S-length.

The subdigraph of a digraph D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$, or $\langle A\rangle$ for brevity. The converse digraph of a digraph D is the digraph obtained from D by reversing all arcs of D.

For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

3. Preliminaries

We now collect the tools which we need in proof of our theorem. In the following, we often use the following definition:

Definition 1: *Let $P = x_1x_2 \dots x_m$ $(m \geq 2)$ be a path in a digraph D and $Q = y_1y_2 \dots y_k$ be a path in $\langle V(D) \setminus V(P)\rangle$ (possibly, $k = 1$). Assume that there is an $i \in [1, m-1]$ such that x_iy_1 and $y_kx_{i+1} \in A(D)$. In this case D contains the path $x_1x_2 \dots x_iy_1y_2 \dots y_kx_{i+1} \dots x_m$ and we say that Q can be inserted into P.*

The following Lemmas 1 and 2 are slightly modified versions of lemma by Häggkvist and Thomas [17] and of lemma by Bondy and Thomassen [9], respectively (their proofs are not too difficult). They will be used extensively in the proof of our result.

Lemma 1: *Let $C_k := x_1x_2 \dots x_kx_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D. Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $\langle V(D) \setminus V(C_k)\rangle$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k+1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$.*

Lemma 2: *Let $P := x_1x_2 \dots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D. Moreover, assume that there exists a path $Q := y_1y_2 \dots y_r$, $r \geq 1$, in $\langle V(D) \setminus V(P)\rangle$. If*

$$d^-(y_1, P) + d^+(y_r, P) \geq k + d^-(y_1, \{x_k\}) + d^+(y_r, \{x_1\}),$$

then there is an $i \in [1, k-1]$ such that x_iy_1 and $y_rx_{i+1} \in A(D)$, i.e., D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$, i.e., Q can be inserted into P. □

The following lemma from [5] is a slightly modified version of Multi-Insertion Lemma due to Bang-Jensen, Gutin and H. Li (see [16], Lemma 5.6.20).

Lemma 3: (H. Li, E. Flandrin. J. Shu [5]). *Let D be a digraph and let P be an (a, b)-path in D. Let Q be a path in $\langle V(D) \setminus V(P)\rangle$ and S be a subset of $V(Q)$. If every vertex of S can be inserted into P, then there exists an (a, b)-path R such that $V(P) \cup S \subseteq V(R) \subseteq V(P) \cup V(Q)$.*

The following lemma also was proved in [5].

Lemma 4: (H. Li, E. Flandrin. J. Shu [5]). *Let D be a digraph of order n and $S \subset V(D)$, $S \neq \emptyset$ be a Meyniel set. Assume that D is S-strongly connected and C is a cycle in D of maximum S-length. If s is an S-vertex of $V(D) \setminus V(C)$, then D contains a C-bypass through s.*

By the inspection of the proof in [12] one can state Lemma 4 in the following form (its proof is the same as the proof of Lemma 4).

Lemma 5: *Let D be a digraph of order n and C be a non-Hamiltonian cycle in D. Let x be an arbitrary vertex not on this cycle C and D contain no C-bypass through x. If in D there are (C,x)- and (x,C)-paths, then the following holds:*

(i). If x is adjacent to some vertex y of C, then D is not 2-strong, $d(x, V(C)\setminus\{y\}) = 0$ and $d(x)+d(z)\leq 2n-2$ for all vertices $z\in V(C)\setminus\{y\}$.

(ii). Assume that x and any vertex of C are nonadjacent, i.e., $d(x,V(C)) = 0$. Let P be a shortest (C,x)-path with $\{u\} = V(P)\cap V(C)$ and Q be a shortest (x,C)-path with $\{v\} = V(Q)\cap V(C)$ (possibly, $u = v$). Then $d(x)+d(z)\leq 2n-2$ for all vertices $z\in V(C)$, maybe except one from $\{u,v\}$. □

In [13], the following was proved:

Lemma 6: (Y. Manoussakis [13]). *Let D be a digraph of order n and $V(D)$ satisfy condition A_0. Assume that there are two distinct pairs of nonadjacent vertices x,y and x,z in D. Then either $d(x)+d(y)\geq 2n-1$ or $d(x)+d(z)\geq 2n-1$.*

It is not difficult to show that we can state Lemma 6 in the following much stronger form:

Lemma 7: *Let D be a digraph of order n and Y be a subset of $V(D)$. Assume that Y satisfies condition A_0 and contains two distinct pairs of nonadjacent vertices x,y and x,z. Then either $d(x)+d(y)\geq 2n-1$ or $d(x)+d(z)\geq 2n-1$.*

For the proof of our result we also need the following simple lemma.

Lemma 8: *Let D be a digraph of order n. Assume that $xy\notin A(D)$ and the vertices x, y in D satisfy the degree condition $d^+(x)+d^-(y)\geq n-2+k$, where $k\geq 1$. Then D contains at least k internally disjoint (x,y)-paths of length two.*

The following lemma also was proved in [5].

Lemma 9: (H. Li, E. Flandrin. J. Shu [5]). *Let D be a digraph of order n and $S\subseteq V(D)$, $S\neq\emptyset$ be a Meyniel set. If D is S-strongly connected, then any two S-vertices s and s' are contained in a cycle of D such that they are at distance at most two on this cycle.*

We can state Lemma 9 in the following form.

Lemma 10: *Let D be a digraph of order n. Assume that a pair of distinct vertices x,y in D satisfies the degree condition $d(x)+d(y)\geq 2n-1$. If D is $\{x,y\}$-strongly connected, then the vertices x and y are contained in a cycle of D such that they are at distance at most two on this cycle.*

Now we will prove the following lemma.

Lemma 11: *Let D be a digraph of order n and Y be a subset of vertices of D with $|Y|\geq 4$. Assume that D is Y-strongly connected and the subset Y satisfies condition A_0. If C is a non-Hamiltonian cycle in D which contains at least two Y-vertices and $y\in V(D)\setminus V(C)$ is an arbitrary Y-vertex, then D contains a C-bypass through y.*

Proof of Lemma 11. If the cycle C contains at least three Y-vertices, then the lemma immediately follows from Lemmas 5 and 7. We may therefore assume that C contains exactly two Y-vertices, say x and u, and there exists a Y-vertex, say y, in $B := V(D)\setminus V(C)$ such that in D there is no C-bypass through y. From $|Y|\geq 4$ it follows that B contains at least two Y-vertices. Let z be an arbitrary Y-vertex in B other than y.

We will consider two cases depending upon the value of $d(y,C)$.

Case 1: $d(y,C)\geq 1$.

Without loss of generality, assume that the vertex y is adjacent to a vertex w of $V(C)$. If $w\notin\{u,x\}$, then from Lemma 5(i) it follows that y,u and y,x are distinct pairs of nonadjacent vertices of Y, $d(y)+d(u)\leq 2n-2$ and $d(y)+d(x)\leq 2n-2$, which contradicts

Lemma 7. We may therefore assume that $w \in \{x, u\}$, for example, let $w = u$ and $yu \in A(D)$. Since we assumed that D had no C-bypass through y, by Lemma 5(i) we have

$$d(y) + d(x) \leq 2n - 2 \quad \text{and} \quad d(y, V(C) \setminus \{u\}) = 0. \tag{1}$$

Now we consider two subcases $xz \in A(D)$, $xz \notin A(D)$.

Subcase 1.1. $xz \in A(D)$.

Then $zy \notin A(D)$ (for otherwise, $xzyu$ is a C-bypass through y, which contradicts our assumption that D has no C-bypass through y). Therefore, the triple of Y-vertices x, y, z satisfies condition A_0, i.e.,

$$d(y) + d(x) + d^-(y) + d^+(z) \geq 3n - 2.$$

This together with $d(y) + d(x) \leq 2n - 2$ (by (1)) imply that $d^+(z) + d^-(y) \geq n$. Hence, by Lemma 8, $z \to v \to y$ for some vertex v other than u. From $d(y, V(C) \setminus \{u\}) = 0$ (by (1)) it follows that $v \in B$. Thus, $xzvyu$ is a C-bypass through y, a contradiction.

Subcase 1.2. $xz \notin A(D)$.

Then by condition A_0 we have

$$d(y) + d(x) + d^+(x) + d^-(z) \geq 3n - 2.$$

Therfore, by (1), $d^+(x) + d^-(z) \geq n$, and hence by Lemma 8 and $xy \notin A(D)$, there exists a vertex v other than u and y such that $x \to v \to z$. It is easy to see that $vy \notin A(D)$ and $zy \notin A(D)$. In this subcase, again we have that $d^+(z) + d^-(y) \geq n$. Hence, by Lemma 8, $z \to a \to y$ for some vertex a other than u and v. By (1), $a \notin V(C)$. Therefore, $a \in B \setminus \{y, z, v\}$ and $vzayu$ or $xvzayu$ is a C-bypass through y when $v \in C$ or not, respectively, a contradiction. The discussion of Case 1 is completed.

Case 2: $d(y, C) = 0$.

By Lemma 5(ii), we have either $d(y) + d(x) \leq 2n - 2$ or $d(y) + d(u) \leq 2n - 2$. Without loss of generality, assume that

$$d(y) + d(x) \leq 2n - 2. \tag{2}$$

This together with condition A_0 imply that

$$d^+(y) + d^-(u) \geq n \quad \text{and} \quad d^+(u) + d^-(y) \geq n. \tag{3}$$

This together with Lemma 8 imply that there are vertices a and v (possibly, $a = v$) other than z such that $u \to v \to y$ and $y \to a \to u$. Observe that v and a are not on C since $d(y, C) = 0$.

First consider the case $d(z, V(C) \setminus \{u\}) \neq 0$. Without loss of generality, assume that $w \in V(C) \setminus \{u\}$ and $zw \in A(D)$ (for the case $wz \in A(D)$ we will consider the converse digraph of D). If $yz \in A(D)$, then $uvyzw$ is a C-bypass through y, a contradiction. We may therefore assume that $yz \notin A(D)$. Then from (2) and condition A_0 it follows that $d^+(y) + d^-(z) \geq n$. Therefore, by Lemma 8, for some vertex $b \in B \setminus \{v\}$, $y \to b \to z$, and hence, $uvybzw$ is a C-bypass through y, which is a contradiction.

Now consider the case $d(z, V(C) \setminus \{u\}) = 0$. Then the vertices z and x are nonadjacent. From (2) and condition A_0 it follows that

$$d^+(x) + d^-(z) \geq n \quad \text{and} \quad d^+(z) + d^-(x) \geq n.$$

From $d^+(z) + d^-(x) \geq n$ and Lemma 8 it follows that there are at least two (z, x)-paths ofthe length two.

Subcase 2.1. There is a (z, x)-paths of the length two, say $z \to b \to x$, such that $b \notin \{u, v\}$.

In this subcase, $yz \notin A(D)$ and $yb \notin A(D)$ (for otherwise, $uvyzbx$ or $uvybx$ is a C-bypass through y, for $yz \in A(D)$ and for $yb \in A(D)$, respectively). Since x, y, z are Y-vertices, from condition A_0 and (2) it follows that $d^+(y) + d^-(z) \geq n$. Now using Lemma 8 and the facts that $yz \notin A(D)$ and $yb \notin A(D)$, we obtain that there exists a vertex $q \in B \setminus \{v, b, z\}$ such that $y \to q \to z$. Thus, $uvyqzbx$ is a C-bypass through y, a contradiction.

Subcase 2.2. There is no $w \in B \setminus \{v\}$ such that $z \to w \to x$.

Then from Lemma 8 and $d^+(z) + d^-(x) \geq n$ it follows that $d^+(z) + d^-(x) = n$ and $zv, vx, zu \in A(D)$ (i.e., there are exactly two (z, x)-paths of the length two). Now using the inequality $d^+(u)+d^-(y) \geq n$ (by (3)) and Lemma 8 we conclude that there exist at least two (u, y)-paths of the length two. If there is a path $u \to c \to y$ such that c is other than v and z, then we may consider the paths $u \to c \to y$ and $z \to v \to x$. For these paths we have the above considered case ($b \notin \{u, v\}$). We may therefore assume that there is no $c \in B \setminus \{v, z\}$ such that $u \to c \to y$. From this and Lemma 8 it follows that $d^+(u) + d^-(y) = n$ and $u \to z \to y$ since $d^+(u) + d^-(y) \geq n$ (by (3)). This together with $vx \in A(D)$ imply that $yv \notin A(D)$ (if $yv \in A(D)$, then $uzyvx$ is a C-bypass through y). Now by condition A_0 and (2) we have

$$d(y) + d(x) = 2n - 2,$$

since x, y, u are Y-vertices, x, y are not adjacent and $uy \notin A(D)$. The last equality implies that $d^+(y) + d^-(x) \geq n - 1$ or $d^-(y) + d^+(x) \geq n - 1$.

If $d^+(y)+d^-(x) \geq n-1$, then, since $yv \notin A(D)$ and $d(y, C) = 0$, from Lemma 8 it follows that there exists a vertex $z_1 \in B\setminus\{v\}$ such that $y \to z_1 \to x$. Therefore, $uvyz_1x$ is a C-bypass through y, a contradiction. We may therefore assume that $d^+(y) + d^-(x) \leq n - 2$. Then $d^-(y)+d^+(x) \geq n$ since $d(y)+d(x) = 2n-2$. Therefore, since $d(y, C) = d(z, V(C)\setminus\{u\}) = 0$, by Lemma 8 there exists a vertex $z_2 \in B \setminus \{a\}$ such that $x \to z_2 \to y$. Hence, xz_2yau is a C-bypass through y, which is a contradiction. This contradiction completes the proof of Lemma 11. □

4. Proof of the Main Result

For readers convenience, again we will formulate the main result.

Theorem 1: *Let D be a digraph of order n and let Y be a nonempty subset of the vertices of D, where $|Y| \geq 2$. Suppose that D is Y-strongly connected and the subset Y satisfies condition A_0. Then D contains a cycle through all the vertices of Y maybe except one.*

Proof: If $|Y| = 2$, then there is nothing to prove. If $|Y| = 3$, then from Y-strongly connectedness of D it follows that Y is an independent set. By Lemma 7, for some two vertices of Y, say x and y, we have $d(x) + d(y) \geq 2n - 1$. Therefore, by Lemma 10, there is a cycle in D that contains the vertices x and y. In the sequel, assume that $|Y| \geq 4$. Now suppose that the subset Y of the vertices of D satisfies the supposition of the theorem but any cycle in D does not contain at least two Y-vertices. By Manoussakis' theorem, we may assume that $Y \neq V(D)$. Since D is Y-strongly connected, using Lemmas 7 and 10 we obtain that in D there exists a cycle which contains at least two Y-vertices. If C is a non-Hamiltonian cycle in D which contains at least two Y-vertices and $y \in V(D) \setminus V(C)$ is an arbitrary Y-vertex, then from Lemma 11 it follows that D contains a C-bypass through

y. In D we choose a cycle C and a C-bypass P_0 through a Y-vertex of $V(D) \setminus V(C)$ such that

(a) C contains as many vertices of Y as possible (C contains at least two Y-vertices),

(b) the gap of C-bypass P_0 is minimum, subject to (a) (by Lemma 11, for the cycle C there exists a C-bypass through any Y-vertex not on C), and

(c) the length of C-bypass P_0 is minimum, subject to (a) and (b).

In the sequel we assume that the cycle $C := x_1x_2 \dots x_mx_1$ and the C-bypass $P_0 := x_1z_1 \dots z_kyz_{k+1} \dots z_tx_{a+1}$ satisfy the conditions (a)-(c), where $1 \leq a \leq m-1$ and $y \in V(D) \setminus V(C)$ is a Y-vertex (possibly, $z_1 = y$ or $y = z_t$). Since the cycle C has the maximum Y-length, it follows that $a \geq 2$ and $C[x_2, x_a]$ contains a Y-vertex. Note that the gap of C-bypass P_0 is equal to a.

Denote $P := z_1 \dots z_kyz_{k+1} \dots z_t$, $E := P[z_1, z_k]$ and $L := P[z_{k+1}, z_t]$. Since the gap a is minimal, the vertex y is not adjacent to any vertex of $C[x_2, x_a]$, i.e, $d(y, C[x_2, x_a]) = 0$. Therefore, by Lemma 2,

$$d(y, C) = d(y, C[x_{a+1}, x_1]) \leq m - a + d^-(y, \{x_1\}) + d^+(y, \{x_{a+1}\}), \tag{4}$$

since any Y-vertex of $B := V(D) \setminus V(C)$ cannot be inserted into C. From the minimality of the path P it follows that

$$d(y, V(P)) \leq |V(P)| + 1 - d^-(y, \{x_1\}) - d^+(y, \{x_{a+1}\}). \tag{5}$$

Notice that $|C[x_2, x_a]| = a - 1$ and $|C[x_{a+1}, x_1]| = m - a + 1$. Firstly for the cycle C and C-bypass $P_0 := x_1z_1 \dots z_kyz_{k+1} \dots z_tx_{a+1}$ we prove the following two claims.

Claim 1: *There is a Y-vertex, say y_1, in $C[x_2, x_a]$ such that y, y_1 are nonadjacent, $d(y) + d(y_1) \leq 2n - 2$ and y_1 cannot be inserted into $C[x_{a+1}, x_1]$. Moreover,*

(i). The path P contains exactly one Y-vertex, namely only y.

(ii). Any Y-vertex of $C[x_2, x_a]$ other than y_1 can be inserted into $C[x_{a+1}, x_1]$.

(iii). There are three (x_{a+1}, x_1)-paths, say P_1, P_2 and P_3, with vertex set $C[x_{a+1}, x_1] \cup F_1$, $C[x_{a+1}, x_1] \cup F_2$ and $C[x_{a+1}, x_1] \cup F_3$, respectively, where $F_1 \subseteq C[x_2, x_a]$, $F_2 \subseteq C[x_2, y_1^-]$, $F_3 \subseteq C[y_1^+, x_a]$ (if $y_1 = x_2$, then $C[x_2, y_1^-] = \emptyset$, if $y_1 = x_a$, then $C[y_1^+, x_a] = \emptyset$) and F_1 (respectively, F_2, F_3) contains all the Y-vertices of $C[x_2, x_a] \setminus \{y_1\}$ (respectively, all the Y-vertices of $C[x_2, y_1^-]$, all the Y-vertices of $C[y_1^+, x_a]$).

Proof: Since we assumed that the cycle C has maximum Y-length, from Lemma 3 it follows that some Y-vertex, say y_1, of $C[x_2, x_a]$ cannot be inserted into $C[x_{a+1}, x_1]$. Hence, using Lemma 2, we obtain

$$d(y_1, C) = d(y_1, C[x_2, x_a]) + d(y_1, C[x_{a+1}, x_1]) \leq 2a - 4 + m - a + 2 = m + a - 2. \tag{6}$$

From the minimality of $C[x_2, x_a]$ it follows that the vertices y and y_1 are nonadjacent.

Put $R := V(D) \setminus (V(C) \cup V(P))$. Now we want to compute the sum of degree y and y_1. By minimality of $C[x_2, x_a]$ we have

$$d(y_1, R) + d(y, R) \leq 2|R| \quad \text{and} \quad d(y, C[x_2, x_a]) = 0. \tag{7}$$

From the minimality of $C[x_2, x_a]$ also it follows that

$$d^+(y_1, \{z_1, \dots, z_k\}) = d^-(y_1, \{z_{k+1}, \dots, z_t\}) = 0. \tag{8}$$

Therefore, $d(y_1,P) \le |P|-1$ since y and y_1 are nonadjacent. This together with the above inequalities (4)-(7) give

$$d(y)+d(y_1)=d(y,R)+d(y_1,R)+d(y,P)+d(y,C)+d(y_1,P)+d(y_1,C)$$
$$\le 2|R|+2|P|+2m-2=2n-2.$$

Thus, $d(y)+d(y_1) \le 2n-2$ for any Y-vertex y of P and for any Y-vertex y_1 of $C[x_2,x_a]$ which cannot be inserted into $C[x_{a+1},x_1]$. This together with Lemma 7 imply that P contains only one Y-vertex, namely y, and any Y-vertex of $C[x_2,x_a]$ different from y_1 can be inserted into $C[x_{a+1},x_1]$. From this and Lemma 3 it immediately follows the third assertion of the claim. Claim 1 is proved. □

By Claim 1, we have

$$d(y_1)+d(y) \le 2n-2. \tag{9}$$

Claim 2: *Let y_1 be a Y-vertex of $C[x_2,x_a]$ which cannot be inserted into $C[x_{a+1},x_1]$. Then $d(y_1,P)=0$.*

Proof: Suppose, on the contrary, that $d(y_1,P) \ge 1$. Then from (8) it follows that either $z_iy_1 \in A(D)$ or $y_1z_j \in A(D)$, for some $i \in [1,k]$ or $j \in [k+1,t]$, respectively. Let $y_1z_j \in A(D)$. We consider the cycle $C_1 := P_3C[x_1,y_1]P[z_j,z_t]x_{a+1}$. This cycle contains all the Y-vertices of C, and hence, has maximum Y-length. It is easy to see that $Q := x_1z_1\dots z_kyz_{k+1}\dots z_j$ is a C_1-bypass through y. By the choice of the cycle C and C-bypass P_0 we have that $y_1=x_a$, i.e., the C-gap of P_0 and C_1-gap of Q are equal but the path $z_1\dots z_kyz_{k+1}\dots z_{j-1}$ is shorter than the path $z_1\dots z_kyz_{k+1}\dots z_t$, which contradicts (c). Therefore, $y_1z_j \notin A(D)$ for all $j \in [k+1,t]$. Similarly, we can prove that $z_iy_1 \notin A(D)$ for all $i \in [1,k]$. Thus,

$$d^-(y_1,\{z_1,\dots,z_k\})=d^+(y_1,\{z_{k+1},\dots,z_t\})=0$$

which together with (8) imply that $d(y_1,P)=0$. Claim 2 is proved. □

Let x be an arbitrary Y-vertex in $B=V(D)\setminus V(C)$ other than y. Claim 1(i) implies that x is not on P. We distinguish two cases according as in $\langle B\setminus(P\setminus\{y\})\rangle$ there exists a path with end-vertices x and y or not.

Case 1: *In $\langle B\setminus(P\setminus\{y\})\rangle$ there exists a path from x to y or there exists a path from y to x.*

Without loss of generality, we may assume that in $\langle B\setminus(P\setminus\{y\})\rangle$ there is an (x,y)-path (for otherwise, we consider the converse digraph of D).

Let H be a shortest (x,y)-path in $\langle B\setminus(P\setminus\{y\})\rangle$. Observe that $x_1x \notin A(D)$, since otherwise, if $x_1x \in A(D)$, then the path P_1 (Claim 1(iii)) together with the arc x_1x and the paths H and $P_0[y,x_{a+1}]$ forms a cycle, say C_1, which contains more Y-vertices than C, which contradicts our assumption that C has maximum Y-length (C_1 contains all Y-vertices of C, except y_1, but contains Y-vertices x and y).

Put $R_0 := B\setminus(V(P)\cup V(H))$ and $H' := H[x_H^+,y_H^-]$ (if $x_H^+=y$, then $H'=\emptyset$).

From the minimality of the gap a (or of the existence of the path P_3) it follows that $y_1x \notin A(D)$ (therefore either $xy_1 \in A(D)$ or x and y_1 are nonadjacent) and

$$d^+(y_1,R_0)+d^-(x,R_0) \le |R_0|. \tag{10}$$

Subcase 1.1. $xy_1 \in A(D)$.

From Lemma 2 it follows that

$$d^-(x,P_1)+d^+(y_1,P_1) \le |P_1|. \tag{11}$$

since $x_1x \notin A(D)$ and the arc xy_1 cannot be inserted into P_1 (for otherwise, D contains a cycle which contains all Y-vertices of C and Y-vertices x, y, which is a contradiction). From the minimality of the gap a, the existence of the paths P_2, P_3 (by Claim 1) and Claim 2 it follows that

$$d^+(y_1, P \cup H) = d^-(x, C[x_2, x_a]) = d^-(x, P) = 0. \quad (12)$$

Clearly,

$$d^+(y_1, S) \leq |S| - 1 \quad \text{and} \quad d^-(x, H') \leq |H'|, \quad (13)$$

where $S := C[x_2, x_a] - P_1$. By adding the above relations (10)-(13), we obtain

$$d^+(y_1) + d^-(x) = d^+(y_1, R_0) + d^-(x, R_0) + d^-(x, P_1) + d^+(y_1, P_1) + d^+(y_1, S) + d^+(y_1, P \cup H)$$

$$+ d^-(x, S) + d^-(x, H') + d^-(x, P) \leq |R_0| + |P_1| + |S| + |H'| - 1 \leq n - 2.$$

This together with (9) give

$$d(y) + d(y_1) + d^-(x) + d^+(y_1) \leq 3n - 4,$$

which contradicts condition A_0, since y, y_1 and x are Y-vertices, y, y_1 are nonadjacent and $y_1x \notin A(D)$.

Subcase 1.2. The vertices x and y_1 are nonadjacent.

We will distinguish two subcases, according as there exists a (y, x)-path in $\langle B \setminus (P \setminus \{y\}) \rangle$ or not.

Subcase 1.2.1. In $\langle B \setminus (P \setminus \{y\}) \rangle$ there is no (y, x)-path, in particular $yx \notin A(D)$.

Then, clearly

$$d^+(y, R_0 \cup H') + d^-(x, R_0 \cup H') \leq |R_0 \cup H'| = |R_0| + |H'|. \quad (14)$$

It is not difficult to see that the path H cannot be inserted into C. Hence, from Lemma 1 it follows that

$$d^-(x, C) + d^+(y, C) \leq |C|. \quad (15)$$

Moreover,

$$d^-(x, P) = d^-(x, E) + d^-(x, L) \leq |L|, \quad \text{since} \quad d^-(x, E) = 0,$$

recall that $E := P[z_1, z_k]$ and $L := P[z_{k+1}, z_t]$, and by the minimality of P, we have

$$d^+(y, P) = d^+(y, E) + d^+(y, L) \leq |E| + 1, \quad \text{since} \quad d^+(y, L) \leq 1.$$

Hence,

$$d^-(x, P) + d^+(y, P) \leq |L| + |E| + 1 = |P|.$$

The last inequality together with (14), (15) and (9) imply that

$$d(y) + d(y_1) + d^-(x) + d^+(y) \leq 2n - 2 + |R| + |H'| + |C| + |P| = 3n - 3,$$

which contradicts condition A_0, since y, y_1 and x are Y-vertices, y, y_1 are nonadjacent and $yx \notin A(D)$.

Subcase 1.2.2. In $\langle B \setminus (P \setminus \{y\}) \rangle$ there is a (y, x)-path.

First consider the case when in $\langle B \setminus (P \cup H \setminus \{x, y\}) \rangle$ there is a (y, x)-path. Let Q be a shortest (y, x)-path in $\langle B \setminus (P \cup H \setminus \{x, y\}) \rangle$.

Let $R_1 := B \setminus (P \cup H \cup Q)$. We want to compute the degree sum of the vertices x and y_1. From the minimality of the $C[x_2, x_a]$ and the existence of the paths H and Q it follows that

$$d(x, R_1) + d(y_1, R_1) \leq 2|R_1| \quad \text{and} \quad d(y_1, H' \cup Q') = 0, \tag{16}$$

where $Q' := Q[y_Q^+, x_Q^-]$ (here if $y_Q^+ = x$, then $Q' = \emptyset$). By Claim 2, $d(y_1, P) = 0$. This and (6) imply that

$$d(y_1, C \cup P) \leq m + a - 2. \tag{17}$$

Now we consider the vertex x. It is not difficult to see that x cannot be inserted into P (for otherwise there exists a (z_1, z_t)-path with vertex set $V(P) \cup \{x\}$ which together with the arcs x_1z_1, z_tx_{a+1} and the path P_1 (Claim 1) forms a cycle which contains all the Y-vertices of C except y_1 but contains Y-vertices x and y, this contradicts the assumption that C has the maximum Y-length). Therefore, by Lemma 2,

$$d(x, P) \leq |P| + 1. \tag{18}$$

From the minimality of the paths H and Q it follows that

$$d(x, Q') \leq |Q'| + 1 \quad \text{and} \quad d(x, H') \leq |H'| + 1. \tag{19}$$

Since the gap a is minimal, we obtain that $d(x, C[x_2, x_a]) = 0$. Using the path P_1 (Claim 1), it is not difficult to see that $x_1x \notin A(D)$ and $xx_{a+1} \notin A(D)$. Therefore, by Lemma 2, $d(x, C) = d(x, C[x_{a+1}, x_1]) \leq m - a$, since x cannot be inserted into C. Summing the above inequalities (16)-(19) and the last inequality, an easy computation shows that

$$d(y_1) + d(x) \leq 2|R_1| + 2m + |P| + |Q'| + |H'| + 1 \leq 2n - 2.$$

This together with $d(y_1) + d(y) \leq 2n - 2$ (by (9)) contradict Lemma 7, since y_1, x and y_1, y are two distinct pairs of nonadjacent vertices in Y.

Now consider the case when any (y, x)-path in $\langle B \setminus (P \setminus \{y\})\rangle$ has a common internal vertex with (x, y)-path H. Then, in particular, the vertices y and x are nonadjacent. Let T be a shortest (y, x)-path in $\langle B \setminus (P \setminus \{y\})\rangle$.

Denote $T' := T[y_T^+, x_T^-]$ and $R_2 := B \setminus (P \cup H' \cup T' \cup \{x\})$. Observe that $|H'| \geq 1$ and $|T'| \geq 1$ since y and x are nonadjacent.

Now we want to compute the sum $d^+(x) + d^-(y)$. It is easy to see that

$$d^+(x, R_2) + d^-(y, R_2) \leq |R_2|, \tag{20}$$

since for otherwise in $\langle B \setminus (P \setminus \{y\})\rangle$ there exists a minimal (y, x)-path which has no common internal vertex with the T. Observe that (y, x)-path T cannot be inserted into $C[x_{a+1}, x_1]$ (for otherwise in D there is a cycle which contains more Y-vertices than the cycle C). Notice that $xx_{a+1} \notin A(D)$ (for otherwise, if $xx_{a+1} \in A(D)$, then the paths $P[z_1, y]$, T and P_1 (Claim 1) and the arcs xx_{a+1}, x_1z_1 form a cycle which has more Y-length than the cycle C). Therefore, by Lemma 2 we have

$$d^+(x, C[x_{a+1}, x_1]) + d^-(y, C[x_{a+1}, x_1]) \leq m - a + d^-(y, \{x_1\}) + d^+(x, \{x_{a+1}\}) \leq m - a + 1. \tag{21}$$

From the minimality of $C[x_2, x_a]$ (i.e., of the gap a) and the existence of the path T it follows that

$$d^-(y, C[x_2, x_a]) = d^+(x, C[x_2, x_a]) = 0. \tag{22}$$

By the minimality of P we have

$$d^-(y,P) = d^-(y,E) + d^-(y,L) \leq |L| + 1.$$

It is not difficult to see that

$$d^+(x,P) = d^+(x,E) + d^+(x,L) \leq |E|,$$

since if $xz_j \in A(D)$ for some $j \in [k+1,t]$, then using the paths T, P_1 and the subpaths $P[z_1,y]$, $P[z_j,z_t]$ we can obtain a cycle which contains more Y-vertices than C. The last two inequalities imply that

$$d^-(y,P) + d^+(x,P) \leq |L| + |E| + 1 = |P|. \tag{23}$$

It remains to compute $d^+(x,H' \cup T')$ and $d^-(y,H' \cup T')$. Denote $T'' := T' \setminus H'$. From the minimality of the path H it follows that $d^+(x,H') = d^-(y,H') = 1$.

This together with the above expressions (20)-(23) imply that

$$d^+(x)+d^-(y) = d^+(x,R_2)+d^-(y,R_2)+d^+(x,C[x_{a+1},x_1])+d^-(y,C[x_{a+1},x_1])+d^+(x,C[x_2,x_a])$$
$$+d^-(y,C[x_2,x_a]) + d^+(x,P) + d^-(y,P) + d^+(x,H') + d^-(y,H') + d^+(x,T'') + d^-(y,T'')$$
$$\leq |R_2| + m - a + 1 + |P| + 2 + d^+(x,T'') + d^-(y,T'')$$
$$= |R_2| + |C| + |P| + 3 + d^+(x,T'') + d^-(y,T'') - a. \tag{24}$$

Assume that $|H'| \geq 2$, then $d^+(x,T'')+d^-(y,T'') \leq |T''|$, since otherwise in $\langle B \setminus (P \setminus \{y\}) \rangle$ there is an (x,y)-path shorter than H. The last inequality together with (24) give

$$d^+(x) + d^-(y) \leq |R_2| + |C| + |P| + 3 + |T''| - a + |H'| - |H'|$$
$$= n + 2 - a - |H'| \leq n - 2,$$

since $a \geq 2$ and $|H'| \geq 2$. This together with (9) imply that

$$d(y) + d(y_1) + d^-(y) + d^+(x) \leq 3n - 4, \tag{25}$$

which contradicts condition A_0 since x, y, y_1 are Y-vertices, y, y_1 are nonadjacent and $xy \notin A(D)$.

Now assume that $|H'| = 1$. We may assume that $|T'| = 1$ (for otherwise, we consider the converse digraph of D). It follows that $T'' = \emptyset$. Now using (9), (24), $a \geq 2$ and $|H'| \geq 1$, we see that again (25) is true, which is a contradiction. The discussion of Case 1 is completed.

Case 2: *In $\langle B \setminus (P \setminus \{y\}) \rangle$ there is no path between the vertices x and y. In particular, x and y are nonadjacent.*

Let $R_3 := B \setminus (P \cup \{x\})$. Then it is easy to see that

$$d(x,R_3) + d(y,R_3) \leq 2|R|,$$

since in $\langle B \setminus (P \setminus \{y\}) \rangle$ there is no path between x and y. Using Lemmas 1 and 2 we obtain that

$$d(x,P) \leq |P| + 1 \quad \text{and} \quad d(x,C) \leq m,$$

since the vertex x can be inserted neither into P nor in C. The last three inequalities together with (4) and (5) imply that

$$d(y) + d(x) \leq 2|R_3| + |P| + m + m - a + |P| + 2 \leq 2n - a \leq 2n - 2.$$

This together with (9) contradict Lemma 7, since $\{x,y\}$ and $\{y,y_1\}$ are two distinct pairs of nonadjacent vertices of Y. The discussion of Case 2 is completed and with it the proof of the theorem is also completed. □

5. Concluding Remarks

Observe that the example of the digraph in Remark 1 is not 2-strongly connected and $|Y| = 3$. We believe that the following may be true.

Conjecture 1: *Let D be a digraph of order $n \geq 4$ and let Y be a nonempty subset of vertices of D which satisfies condition A_0. Then D has a cycle that contains all the vertices of Y if either (i) or (ii) or (iii) below is satisfied:*

(i) D is 2-strongly connected.

(ii) D is Y-strongly connected and $|Y| \geq 4$.

(iii) for any ordered pair of distinct vertices x, y of Y there are two internally disjoint paths from x to y in D.

C. Thomassen [18] (for $n = 2k+1$) and the author [19] (for $n = 2k$) proved the following theorem.

Theorem H: (C.Thomassen [18], S. Darbinyan [19]). *Let D be a digraph of order $n \geq 5$ with minimum degree at least $n-1$ and with minimum semi-degree at least $n/2-1$. Then D is Hamiltonian or belongs to a non-empty finite family of non-Hamiltonian digraphs, which are characterized.* □

A question was put in [20]:

Let D be a digraph of order $n \geq 5$ and let $T \neq \emptyset$ be a subset of $V(D)$. Assume that D is strongly connected (or D is T-strongly connected) and every vertex of T has a degree at least $n-1$ and has an outdegree and an indegree at least $n/2-1$. Does D have a cycle that contains all the vertices of T?.

For $n = 2m+1$ in [20] it was proved:

If D is strongly connected and contains a cycle of length $n-1$, then D has a cycle containing all the vertices of T unless some extremal cases.

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Մանոուսակիսի տիպի պայմանին բավարարող կողմնորոշված գրաֆների ցիկլիկության մասին

Ս. Դարբինյան

Ամփոփում

Ներկա աշխատանքում ցույց է տրվում որ եթե կողմնորոշված գրաֆի գագաթների Y ենթաբազմությունը բավարարում է D կողմնորոշված գրաֆների համար Մանոուսակիսի

hամիլտոնյանության բավարար պայմանին (J. Graph Theory, 16, 1992), ապա այդ գրաֆը պարունակում է ցիկլ, որն անցնում է Y ենթաբազմությանը պատկանող բոլոր գագաթներով բացի գուցե մեկից: Ստացված արդյունքը լուծում է Լիի, Ֆլանդրինի և Շուի (Discrete Mathematics, 307, 2007) կողմից առաջարկած խնդիրը:

О цикличности орграфов при условий типа Маноусакиса

С. Дарбинян

Аннотация

В работе доказано, что если подмножество Y вершин орграфа D удовлетворяет достаточному условию гамильтоновсти Маноусакиса (J. Graph Theory, 16, 1992), то в D существует контур, который содержит по крайней мере $|Y|-1$ вершин подмножества Y . Полученный результат решает задачу Ли, Фландрин и Шу (Discrete Mathematics, 307, 2007).