Mathematical Problems of Computer Science 48, 14–18, 2017.

New Extensions of Dirac's Theorems

Mossine S. Koulakzian[1], Carlen M. Mosesyan[2] and Zhora G. Nikoghosyan[1]

[1]Institute for Informatics and Automation Problems of NAS RA
e-mails: mossine@hotmail.com; zhora@ipia.sci.am
[2]Faculty of Mathematics, Physics and Informatics
Kh. Abovyan Armenian State University
e-mail: mosesyan@list.ru

Abstract

Let G be a graph on n vertices with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$ and let c be the circumference - the length of a longest cycle in G. In 1952, Dirac proved: (i) every graph with $d_1 \geq \frac{n}{2}$ is hamiltonian; (ii) in every 2-connected graph, $c \geq \min\{n, 2d_1\}$. In this paper we present the following two Dirac-type extensions: (iii) every graph with $d_\delta \geq \frac{n}{2}$ is hamiltonian; (iv) in every 2-connected graph, $c \geq \min\{n, 2d_\delta\}$. The results are sharp.

Keywords: Hamilton cycle, Longest cycle, Circumference, Minimum degree, Degree sequence.

1. Introduction

We consider only finite undirected graphs with neither loops nor multiple edges. A good reference for any undefined terms is [2].

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. Let n be the order (the number of vertices) of G, c the order of a longest cycle (called circumference) in G and p the order of a longest path. We use $N(v)$ to denote the set of all neighbors of a vertex v and $d(v) = |N(v)|$ to denote the degree of vertex v. The minimum degree in G is denoted by δ. Let $d_1, d_2, ..., d_n$ be the degree sequence in G with $\delta = d_1 \leq d_2 \leq ... \leq d_n$.

A path (simple path) of order m is a sequence of distinct vertices $v_1, ..., v_m$, denoted by $v_1v_2...v_m$, such that $v_{i-1}v_i$ is an edge for all $2 \leq i \leq m$. Similarly. a cycle of order m is a sequence of distinct vertices $v_1, ..., v_m$, denoted by $v_1v_2...v_mv_1$, such that $v_{i-1}v_i$ and v_mv_1 are edges for all $2 \leq i \leq m$. In particular, for $m = 2$, $v_1v_2v_1$ is a cycle of order 2; and for $m = 1$, v_1v_1 is a cycle of order 1. So, by the definition, every vertex (edge) can be considered as a cycle of order 1 (2, respectively). A graph G is hamiltonian if G contains a Hamilton cycle, that is a simple spanning cycle.

We write a cycle (a path) Q with a given orientation by $\overrightarrow{Q}$. The reverse sequence of vertices of $\overrightarrow{Q}$ is denoted by $\overleftarrow{Q}$. For $x \in V(Q)$, we denote the successor and the predecessor of x on $\overrightarrow{Q}$ (if such vertices exist) by x^+ and x^-, respectively. For $U \subseteq V(Q)$, we denote $U^+ = \{u^+|u \in U\}$ and $U^- = \{u^-|u \in U\}$. We use $P = x\overrightarrow{P}y$ to denote a path with end vertices x and y in the direction from x to y. We say that vertex z_1 precedes vertex z_2 on a path $\overrightarrow{Q}$ if z_1, z_2 occur on $\overrightarrow{Q}$ in this order, and indicate this relationship by $z_1 \prec z_2$.

In 1952, Dirac [3] gave the first sufficient condition for a graph to be hamiltonian and the first nontrivial lower bound for the circumference in terms of minimum degree $d_1 = \delta$.

Theorem A: [3]. *Every graph with* $d_1 \geq \frac{n}{2}$ *is hamiltonian.*

Theorem B: [3]. *In every 2-connected graph,* $c \geq \min\{n, 2d_1\}$.

A great number of generalizations and improvements of Theorems A and B are known under various conditions.

In this paper we present the following two Dirac-type extensions of Theorems A and B in terms of d_i (for appropriate i) instead of d_1.

Theorem 1: *Every graph with* $d_\delta \geq \frac{n}{2}$ *is hamiltonian.*

Theorem 2: *In every 2-connected graph,* $c \geq \min\{n, 2d_\delta\}$.

It is not hard to see that if G is a graph with $d_\delta \geq \frac{n}{2}$ or G is a 2-connected graph, then $\delta \geq 2$.

To see that Theorem 1 and Theorem 2 are sharp, let $G = K_\delta + \overline{K}_{\delta+1}$. Since $d_\delta = \delta$ and $c = 2\delta < n$, we conclude that the bound $d_\delta \geq \frac{n}{2}$ in Theorem 1 cannot be replaced by $d_\delta \geq \frac{n-1}{2}$.

Let $G = K_\delta + (\delta K_1 \cup K_2)$. Clearly, $d_\delta = \delta$ and $d_{\delta+1} = \delta + 1$. Observing also that $c = 2\delta + 1 < n$, we conclude that the bound $d_\delta \geq \frac{n}{2}$ in Theorem 1 cannot be replaced by $d_{\delta+1} \geq \frac{n}{2}$.

As for Theorem 2, the graph example $G = K_\delta + \overline{K}_{\delta+1}$ shows that the bound $c \geq \min\{n, 2d_\delta\}$ in Theorem 2 cannot be replaced by $c \geq \min\{n, 2d_\delta + 1\}$. Finally, the graph example $G = K_\delta + (\delta K_1 \cup K_2)$ shows that the bound $c \geq \min\{n, 2d_\delta\}$ cannot be replaced by $c \geq \min\{n, 2d_{\delta+1}\}$. Thus, Theorem 1 and Theorem 2 are sharp in all respects.

Let $\overrightarrow{P} = v_1v_2...v_p$ be a longest path in G. Clearly, $N(v_1) \cup N(v_p) \subseteq V(P)$. A vine of length m on P is a set

$$\{L_i = w_i \overrightarrow{L}_i z_i : 1 \leq i \leq m\}$$

of internally-disjoint paths such that

(*a*) $V(L_i) \cap V(P) = \{w_i, z_i\}$ $(i = 1, ..., m)$,

(*b*) $v_1 = w_1 \prec w_2 \prec z_1 \preceq w_3 \prec z_2 \preceq w_4 \prec ... \preceq w_m \prec z_{m-1} \prec z_m = v_p$ on P.

Lemma 1: [3]. *If G is 2-connected, then there is at least one vine on P.*

We need also the following lemma.

Lemma 2: [1]. *Let G be a 2-connected graph on n vertices. If $v_1v_2...v_p$ is a longest path of G, then there exists a cycle of length at least* $\min\{d(v_1) + d(v_p), n\}$.

2. Proofs

Proof of Theorem 1. Assume first that G is not connected and let H_1 and H_2 be two connected components of G. Clearly, $|V(H_i)| \geq \delta + 1$ $(i = 1, 2)$ and

$$\max\{d(v) : v \in V(H_i)\} \geq d_{\delta+1} \quad (i = 1, 2).$$

Hence

$$2d_\delta \leq 2d_{\delta+1} \leq \max\{d(v) : v \in V(H_1)\} + \max\{d(v) : v \in V(H_2)\}$$

$$\leq |V(H_1)| + |V(H_2)| - 2 \leq n - 2,$$

contradicting the hypothesis $2d_\delta \geq n$. So, we can assume that G is connected. Let $\overrightarrow{P} = v_1v_2...v_p$ be a longest path in G. Clearly, $N(v_1) \cup N(v_p) \subseteq V(P)$. Assume that

(a1) P is chosen so that $d(v_1)$ is maximum.

Let $x_1, x_2, ..., x_t$ be the elements of $N(v_1)$ occurring on $\overrightarrow{P}$ in a consecutive order, where $t = d(v_1) \geq \delta$. Observe that for each $i \in \{2, ..., t\}$,

$$x_i^- \overleftarrow{P} v_1 x_i \overrightarrow{P} v_p$$

is a longest path in G. By (a1),

$$d(v_1) \geq d(x_i^-) \quad (i = 1, 2, .., t). \tag{1}$$

Assume first that $x_t = v_p$, that is $c \geq p$. If $c \geq p+1$, then the cycle of order at least $p+1$ contains a path of order at least $p+1$, a contradiction. Hence, $c = p$. Put $\overrightarrow{C} = v_1v_2...v_pv_1$. If $p = n$, then C is a Hamilton cycle in G. Otherwise, since G is connected, $u_1u_2 \in E(G)$ for some $u_1 \in V(C)$ and $u_2 \notin V(C)$. Then $u_1^+ \overrightarrow{C} u_1 u_2$ is a path of order $p+1$, a contradiction.

Now assume that $x_t \neq v_p$, that is $x_t \prec v_p$. Further, we can assume that

(a2) P is chosen so that $d(v_p)$ is maximum, subject to (a1).

Let $y_1, y_2, ..., y_f$ be the elements of $N(v_p)$ occurring on $\overleftarrow{P}$ in a consecutive order. By (a2),

$$d(v_p) \geq d(y_i^+) \quad (i = 1, 2, .., f). \tag{2}$$

By (1) and (2),

$$d(v_1) \geq \max\{d(x_1^-), d(x_2^-), ..., d(x_t^-)\}$$
$$\geq \max\{d_1, d_2, ..., d_t\} = d_t = d_{d(v_1)} \geq d_\delta.$$

and

$$d(v_p) \geq \max\{d(y_1^+), d(y_2^+), ..., d(y_f^+\}$$
$$\geq \max\{d_1, d_2, ..., d_f\} = d_f = d_{d(v_p)} \geq d_\delta,$$

implying that

$$d(v_1) + d(v_p) \geq 2d_\delta \geq n. \tag{3}$$

By Lemma 2, G is hamiltonian. However, we present a short proof of this fact according to the latest terminology.

Case 1. $N(v_1) \cap N^+(v_p) \neq \emptyset$.

Let $v \in N(v_1) \cap N^+(v_p)$, that is $v_1v, v_pv^- \in E(G)$. Since

$$v_1 v \overrightarrow{P} v_p v^- \overleftarrow{P} v_1$$

is a cycle of order p, we have $c \geq p$, implying that $c = p$. Since G is connected, we have $c = p = n$, that is G is hamiltonian.

Case 2. $N(v_1) \cap N^+(v_p) = \emptyset$.

It follows that

$$n \geq p \geq |N(v_1)| + |N^+(v_p)| + |\{v_1\}| \geq$$
$$\geq |N(v_1)| + |N(v_p)| + 1 \geq d(v_1) + d(v_p) + 1.$$

By (3), $n \geq 2d_\delta + 1$, contradicting the hypothesis $d_\delta \geq \frac{n}{2}$. ■

Proof of Theorem 2. Let $\overrightarrow{P} = v_1 v_2 ... v_p$ be a longest path in G. Define the vertices $x_1, x_2, ..., x_t, y_1, y_2, ..., y_f$ as in proof of Theorem 1, where we have proved

$$d(v_1) + d(v_p) \geq 2d_\delta. \tag{4}$$

Case 1. $x_t \preceq y_f$.
Let

$$\{L_i = w_i \overrightarrow{L}_i z_i : 1 \leq i \leq m\}$$

be a vine of minimal length m on $\overrightarrow{P}$. Since P is a longest path in G, we have $L_1, L_M \in E(G)$. Next, since m is minimal, we have $x_t \prec z_2$, $x_t \prec w_3$ and $w_{m-1} \prec y_f$, $z_{m-2} \prec y_f$. Choose $z_1^* \in V(P)$ such that $w_2 \prec z_1^*$ and $|V(w_2 \overrightarrow{P} z_1^*)|$ is minimal. Analogously, choose $w_m^* \in V(P)$ such that $w_m^* \prec z_{m-1}$ and $|V(w_m^* \overrightarrow{P} z_{m-1})|$ is minimal. Put

$$H = P \cup \bigcup_{i=2}^{m-1} L_i \cup \{v_1 z_1^*, v_p w_m^*\}.$$

By deleting the following paths

$$w_i \overrightarrow{P} z_{i-1} \quad (i = 3, 4, ..., m-1), \quad w_2 \overrightarrow{P} z_1^*, \quad w_m^* \overrightarrow{P} z_{m-1}$$

from H (except for their end vertices), we obtain a cycle C with at least $d(v_1) + d(v_p) + 1$ vertices. By (4),

$$c \geq |V(C)| \geq d(v_1) + d(v_p) + 1 > 2d_\delta.$$

Case 2. $y_f \prec x_t$.
Case 2.1. $N(v_1) \cap N^+(v_p) \neq \emptyset$.
Let $v \in N(v_1) \cap N^+(v_p)$, that is $v_1 v, v_p v^- \in E(G)$. Since

$$v_1 v \overrightarrow{P} v_p v^- \overleftarrow{P} v_1$$

is a cycle of order p and G is connected, either $p < |V(G)|$ and we can form a path longer than P (a contradiction) or $p = |V(G)|$, implying that $c = p = n$.
Case 2.2. $N(v_1) \cap N^+(v_p) = \emptyset$.
Since $y_f \prec x_t$, we can choose $x_i \in N(v_1)$ and $y_j \in N(v_p)$ such that $y_j \prec x_i$ and $v_1 v, v_p v \notin E(G)$ for each vertex v with $y_j \prec v \prec x_i$. Put

$$C = v_1 x_i \overrightarrow{P} v_p y_i \overleftarrow{P} v_1.$$

Then

$$c \geq |V(C)| \geq |N(v_1)| + |N^+(v_p)| + |\{v_1\}| - |\{y_j\}|$$
$$\geq |N(v_1)| + |N(v_p)| = d(v_1) + d(v_p).$$

By (4), $c \geq 2d_\delta$. ■

3. Acknowledgments

We would like to thank the referees for useful suggestions, which have considerably improved the presentation of the paper.

References

[1] J. A. Bondy, "Basic graph theory - paths and cycles", Handbook of Combinatorics, vol.1, Elsevier, Amsterdam, pp. 5-110, 1995.

[2] J. A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York 1976.

[3] G. A. Dirac, "Some theorems on abstract graphs", *Proc. London, Math. Soc.*, vol. 2, pp. 69-81, 1952.

Submitted 22.08.2017, accepted 04.12.2017.

Դիրակի թեորեմների նոր ընդհանրացումներ

Մ. Քուլաքզյան, Կ. Մոսեսյան և Ժ. Նիկողոսյան

Ամփոփում

Դիցուք G-ն δ նվազագույն աստիճան և $\delta = d_1 \leq d_2 \leq ... \leq d_n$ աստիճանային հաջորդականություն ունեցող n գագաթանի գրաֆ է: G-ի ամենաերկար ցիկլի երկարությունը նշանակվում է c-ով: 1952-ին Դիրակն ապացուցեց, որ (i) $d_1 \geq \frac{n}{2}$ պայմանին բավարարող կամայական գրաֆ ունի Համիլտոնի ցիկլ. (ii) կամայական 2-կապակցված գրաֆում $c \geq \min\{n, 2d_1\}$: Ներկա աշխատանքում բերվում են նշված արդյունքների երկու ընդլայնումներ.(iii) $d_\delta \geq \frac{n}{2}$ պայմանին բավարարող կամայական գրաֆ ունի Համիլտոնի ցիկլ, (iv) կամայական 2-կապակցված գրաֆում $c \geq \min\{n, 2d_\delta\}$: Ստացված արդյունքները ենթակա չեն բարելավման:

Новые обобщения теорем Дирака

М. Кулакзян, К. Мосесян и Ж. Никогосян

Аннотация

Пусть $\delta = d_1 \leq d_2 \leq ... \leq d_n$ последовательность степеней вершин n-вершинного графа G с минимальной степенью δ. Длина длиннейшего цикла графа обозначается через c. В 1952 году Дирак доказал, что (i) каждый граф удовлетворяющий условию $d_1 \geq \frac{n}{2}$, имеет Гамльтонов цикл; (ii) если G является 2-связным графом, то $c \geq \min\{n, 2d_1\}$. В настоящей работе доказываются: (iii) каждый граф удовлетворяющий условию $d_\delta \geq \frac{n}{2}$, имеет Гамльтонов цикл; (iv) если G является 2-связным графом, то $c \geq \min\{n, 2d_\delta\}$. Полученные результаты неулучшаемы.