Mathematical Problems of Computer Science 62, 52–58, 2024.
doi:10.51408/1963-0120

UDC 519.2

On Sub-Gaussianity in Banach Spaces

George J. Giorgobiani, Vakhtang V. Kvaratskhelia and Vazha I. Tarieladze

Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University,
Tbilisi, Georgia
e-mail: giorgobiani.g@gtu.ge, v.kvaratskhelia@gtu.ge, v.tarieladze@gtu.ge

Abstract

We show that if X is a Banach space and a weakly sub-Gaussian random element in X induces the 2-summing operator, then it is $T-$sub-Gaussian, provided that X is a reflexive type 2 space. Using this result, we obtain a characterization of weakly sub-Gaussian random elements in a Hilbert space which are $T-$sub-Gaussian.

Keywords: Sub-Gaussian random variable, Gaussian random variable, weakly sub-Gaussian random element, $T-$sub-Gaussian random element, Banach space, Hilbert space.

Article info: Received 7 October 2024; sent for review 15 October 2024; accepted 26 November 2024.

Acknowledgment: The work was partially supported by the European Union's HORIZON EUROPE Grant Project GAIN.

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Following [8], we call a real-valued measurable function $\xi : \Omega \to \mathbb{R}$ a sub-Gaussian random variable if there exists a real number $a \geq 0$ such that for every real number t the following inequality is valid

$$\mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}a^2t^2},$$

where $\mathbb{E}$ stands for the mathematical expectation.

To each random variable ξ, there corresponds a parameter $\tau(\xi) \in [0, +\infty]$ defined as follows (we agree $\inf(\emptyset) = +\infty$):

$$\tau(\xi) = \inf\left\{a \geq 0 : \quad \mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}a^2t^2}, \quad t \in \mathbb{R}\right\}.$$

A random variable ξ is sub-Gaussian if and only if $\tau(\xi) < +\infty$ and $\mathbb{E}\xi = 0$. Moreover, if ξ is a sub-Gaussian random variable, then for every real number t

$$\mathbb{E}\, e^{t\xi} \leq e^{\frac{1}{2}\tau^2(\xi)t^2}$$

and

$$\left(\mathbb{E}\xi^2\right)^{\frac{1}{2}} \leq \tau(\xi)\,.$$

If ξ is a Gaussian random variable with $\mathbb{E}\xi = 0$, then ξ is sub-Gaussian and

$$\left(\mathbb{E}\xi^2\right)^{\frac{1}{2}} = \tau(\xi)\,.$$

Remark 1.1 *[3, Example 1.2]. If ξ is a bounded random variable, i.e., if for some constant $c \in \mathbb{R}$ with $1 \leq c < +\infty$, we have $|\xi| \leq c$ a.s. and $\mathbb{E}\xi = 0$, then ξ is sub-Gaussian and $\tau(\xi) \leq c$.*

Denote by $\mathcal{SG}(\Omega, \mathcal{A}, \mathbb{P})$, or in short, by $\mathcal{SG}(\Omega)$ the set of all sub-Gaussian random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. $\mathcal{SG}(\Omega)$ is a vector space over $\mathbb{R}$ with respect to the natural point-wise operations; moreover, the functional $\tau(\cdot)$ is a norm on $\mathcal{SG}(\Omega)$ (provided that random variables that coincide almost surely are identified) and $(\mathcal{SG}(\Omega), \tau(\cdot))$ is a Banach space [2]. For $\xi \in \mathcal{SG}(\Omega)$ instead of $\tau(\xi)$ we will write also $\|\xi\|_{\mathcal{SG}(\Omega)}$.

More information about the sub-Gaussian random variables can be found, for example, in [6].

Remark 1.2 *[3, Theorem 1.3] (see also[14, Proposition 2.9]). For a sub-Gaussian random variable ξ, we have*

$$\vartheta(\xi) = \sup_{n\geq 1} \frac{\left(\mathbb{E}\,\xi^{2n}\right)^{1/2n}}{n^{1/2}} < +\infty,$$

the functional ϑ is a norm on the vector space $\mathcal{SG}(\Omega)$ and the norms τ and ϑ are equivalent, i.e., there exist positive constants a_1 and a_2 such that for every $\xi \in \mathcal{SG}(\otimes)$ we have

$$a_1\vartheta(\xi) \leq \tau(\xi) \leq a_2\vartheta(\xi).$$

In an infinite dimensional Banach space there are several notions of sub-Gaussianity. The aim of the paper is to show that these concepts are different in general. We also give some sufficient conditions for their equivalence.

Let X be a Banach space over $\mathbb{R}$ with a norm $\|\cdot\|$ and X^* be its dual space. The value of the linear functional $x^* \in X^*$ at an element $x \in X$ is denoted by the symbol $\langle x^*, x\rangle$.

Following [15, p. 88], a mapping $\xi : \Omega \to X$ is called a random element (vector) in X if $\langle x^*, \xi\rangle$ is a random variable for every $x^* \in X^*$.

If $0 < p < \infty$, then a random element ξ in a Banach space X:

- has *a strong p-th order*, if $\|\xi\|$ is a random variable and $\mathbb{E}\,\|\xi\|^p < \infty$;
- has *a weak p-th order*, if $\mathbb{E}\,|\langle x^*, \xi\rangle|^p < \infty$ for every $x^* \in X^*$;
- is *centered*, if ξ has a weak first order and $\mathbb{E}\,\langle x^*, \xi\rangle = 0$ for every $x^* \in X^*$.

To each weak second-order centered random element ξ in a separable Banach space X, there corresponds a mapping $R_\xi : X^* \to X$ such that

$$\langle y^*, R_\xi x^*\rangle = \mathbb{E}\,\langle y^*, \xi\rangle\langle x^*, \xi\rangle, \quad \text{for every} \quad x^*, y^* \in X^*,$$

which is called *the covariance operator of* ξ [15, Corollary 2 (p.172)].

A random element $\xi : \Omega \to X$ is called *Gaussian*, if for each functional $x^* \in X^*$, the random variable $\langle x^*, \xi \rangle$ is Gaussian.

A mapping $R : X^* \to X$ is said to be *a Gaussian covariance* if there exists a Gaussian random element in X, the covariance operator of which is R.

A random element $\xi : \Omega \to X$ will be called *weakly sub-Gaussian* [13] if for each $x^* \in X^*$, the random variable $\langle x^*, \xi \rangle$ is sub-Gaussian.

A random element $\xi : \Omega \to X$ will be called *$T-$sub-Gaussian* (or $\gamma-$sub-Gaussian [5]) if there exists a probability space $(\Omega', \mathcal{A}', \mathbf{P}')$ and a centered Gaussian random element $\eta : \Omega' \to X$ such that for each $x^* \in X^*$

$$\mathbb{E}\, e^{\langle x^*, \xi \rangle} \le \mathbb{E}\, e^{\langle x^*, \eta \rangle}\,. \tag{1.1}$$

Theorem 1.3 (a) *If X is a finite-dimensional Banach space, then every weakly sub-Gaussian random element in X is $T-$sub-Gaussian.*

(b) *If X is a infinite-dimensional separable Banach space, then there exists a weakly sub-Gaussian random element in X, which is not $T-$sub-Gaussian.*

Proof.

(a) See [14, Proposition 4.9].

(b) According to [13] (see also [14, Theorem 4.5]), we can find and fix a weakly sub-Gaussian random element ξ in X, such that $\mathbb{E}\|\xi\| = \infty$. Such a random element cannot be $T-$sub-Gaussian, because as stated in [5, Theorem 3.4] every such random element must be "exponentially integrable". ■

To every weakly sub-Gaussian random element $\xi : \Omega \to X$, we associate *the induced linear operator*

$$T_\xi : X^* \to \mathcal{SG}(\Omega)$$

defined by the equality:

$$T_\xi x^* = \langle x^*, \xi \rangle \qquad \text{for all} \quad x^* \in X^*.$$

Let X and Y be Banach spaces, $L(X,Y)$ be the space of all continuous linear operators acting from X to Y. An operator $T \in L(X,Y)$ is called 2-(absolutely) summing if there exists a constant $C > 0$ such that for each natural number n and for every choice $x_1, x_2, \ldots, x_n$ of elements from X, we have

$$\left(\sum_{k=1}^{n} ||Tx_k||^2 \right)^{1/2} \le C \sup_{||x^*||_{X^*} \le 1} \left(\sum_{k=1}^{n} |\langle x^*, x_k \rangle|^2 \right)^{1/2} . \tag{1.2}$$

For a 2-summing $T : X \to Y$, we denote the minimum possible constant C in (1.2) by $\pi_2(T)$.

We say that a Banach space X has type 2 if there exists a finite constant $C \ge 0$ such that for each natural number n and for every choice $x_1, x_2, \ldots, x_n$ of elements from X, we have

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^2 dt \right)^{1/2} \le C \left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2},$$

where $r_1(\cdot), \ldots, r_n(\cdot)$ are Rademacher functions on $[0,1]$. An example of a type 2 space is a Hilbert space as well as the spaces $l_p, L_p([0,1]), 2 \le p < +\infty$.

2 Main results

The following theorem is a slightly corrected version of [9, Theorem 1.7].

Theorem 2.1 *Let X be a separable Banach space. For a weakly sub-Gaussian random element $\xi : \Omega \to X$, consider the assertions:*

(i) ξ is $T-$sub-Gaussian.

(ii) $T_\xi : X^* \to \mathcal{SG}(\Omega)$ is a 2-summing operator.

Then:

(a) (i) $\Longrightarrow$ (ii);

(b) The implication (ii) $\Longrightarrow$ (i) is true provided that X is a reflexive Banach space of type 2.

Proof.

(*a*) (*i*) implies that there exists a centered Gaussian random element $\eta : \Omega' \to X$ such that for each $x^* \in X^*$ the relation (1.1) holds. This implies that

$$\tau(T_\xi x^*) \leq \tau(T_\eta x^*) \qquad \text{for all} \quad x^* \in X^* .$$

Thus, as η is a Gaussian random element in X, the operator T_η is 2-summing (see, for example, [4]). Hence, we conclude that (*ii*) holds.

(*b*) Since ξ is a weakly sub-Gaussian random element, for every $x^* \in X^*$, we can write:

$$\mathbb{E}\, e^{\langle x^*, \xi\rangle} \leq e^{\frac{1}{2}||T_\xi x^*||^2_{\mathcal{SG}(\Omega)}}.$$

Taking into account that the operator T_ξ is 2-summing and X is reflexive, by Pietsch domination theorem (see [10] or [15, Theorem 2.2.2]), there exists a probability measure μ defined on the $\sigma(X, X^*)$-Borel sigma-algebra of the unit ball $B_X \subset X$ such that

$$||T_\xi x^*||^2_{\mathcal{SG}(\Omega)} \leq \pi_2^2(T_\xi) \int\limits_{B_X} \langle x^*, x\rangle^2 \,\mu(dx), \quad x^* \in X^* .$$

If we consider μ as a probability measure in X concentrated on B_X, then for every $x^* \in X^*$

$$\int\limits_{B_X} \langle x^*, x\rangle^2 \,\mu(dx) = \int\limits_{X} \langle x^*, x\rangle^2 \,\mu(dx) = \langle R_\mu x^*, x^*\rangle,$$

where R_μ is the covariance operator of μ. As μ is concentrated on the bounded set, it clearly has a strong second order, and taking into account the fact that X is a type 2 space, we obtain that R_μ is a Gaussian covariance (see [4, Theorem 3.1]). Denoting $\pi_2^2(T)R_\mu = R$, we get

$$\mathbb{E}\, e^{\langle x^*, \xi\rangle} \leq e^{\frac{1}{2}\langle R x^*, x^*\rangle}, \qquad x^* \in X^*,$$

and, thus, ξ is a $T-$sub-Gaussian random element as R is a Gaussian covariance. ■

Problem 2.2 *Problem. Prove that the reflexivity condition for X in Theorem 2.1(b) can be removed.*

Consider now the case when $X = H$, where H denotes an infinite-dimensional separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. As usual we identify H^* with H by means of the equality $H^* = \{\langle \cdot, y \rangle : y \in H\}$.

From Theorem 2.1, we will derive now the following result, which is related to a similar assertion contained in [1, Proposition 3.1].

Theorem 2.3 *Let H be an infinite-dimensional separable Hilbert space. For a weakly sub-Gaussian random element $\xi : \Omega \to H$, the following statements are equivalent:*

(i) ξ is $T-$sub-Gaussian.

(ii_m) For each orthonormal basis (φ_k) of H,

$$\sum_{k=1}^{\infty} \tau^2(\langle \varphi_k, \xi \rangle) < \infty \,. \tag{2.1}$$

Proof. The implication $(i) \Longrightarrow (ii_m)$ follows from Theorem 2.1 (a).

The implication $(ii_m) \Longrightarrow (i)$ follows from Theorem 2.1 (b) as H is a type 2 space and according to [11], the condition (ii_m) implies that the condition (ii) of Theorem 2.1 is satisfied as well. ■

In connection with Theorem 2.3, the following question naturally arises: is it possible to replace the condition (ii_m) by the following (weaker) condition?

(ii_w) There is an orthonormal basis (φ_k) of H such that

$$\sum_{k=1}^{\infty} \tau^2(\langle \varphi_k, \xi \rangle) < \infty \,.$$

In [1, Remark 4.3], it is claimed that the answer to this question *is positive.*

At the end, we pose another interesting question related to Theorem 2.3: does there exist a bounded centered random element ξ in a separable infinite-dimensional Hilbert space H such that

$$\sum_{k=1}^{\infty} \tau^2(\langle \psi_k, \xi \rangle) = \infty$$

for *every* orthonormal basis (ψ_k) of H?

3 Conclusion

We have shown that in an infinite dimensional Banach space, the notions of weak sub-Gaussianity and T-sub-Gaussianity do not coincide. Sufficient conditions for their equivalence in a general, infinite-dimensional Banach space is given in terms of 2-summing induced operators.

The work was partially supported by the European Union's HORIZON EUROPE Grant Project GAIN.

We are very grateful to the referees for the comments and the careful reading of the paper.

References

[1] R. G. Antonini, "Subgaussian random variables in Hilbert spaces", *Rend. Sem. Mat. Univ. Padova*, vol. 98, pp. 89–99, 1997.

[2] V. V. Buldygin and Yu. V. Kozachenko, "Sub-Gaussian random variables", *Ukrainian Mathematical Journal*, vol. 32, pp. 483–489, 1980.

[3] V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes.* American Mathematical Soc., 2000.

[4] S. A. Chobanjan and V. I. Tarieladze, "Gaussian characterizations of certain Banach spaces", *J. Multivariate Anal.*, vol. 7, no. 1, pp. 183–203, 1977.

[5] R. Fukuda, "Exponential integrability of sub-Gaussian vectors", *Probab. Theory Relat. Fields*, vol. 85, no. 4, pp. 505–521, 1990.

[6] G. Giorgobiani, V. Kvaratskhelia and M. Menteshashvili, "Unconditional Convergence of Sub-Gaussian Random Series", *Pattern Recognition and Image Analysis*, vol. 34, no. 1, pp. 92–101, 2024.

[7] G. Giorgobiani, V. Kvaratskhelia and V. Tarieladze, "Notes on sub-Gaussian random elements", *In Applications of Mathematics and Informatics in Natural Sciences and Engineering: AMINSE 2019*, Tbilisi, Georgia, pp. 197–203, Springer International Publishing, 2020.

[8] J. P. Kahane, "Proprietes locales des fonctions a series de Fourier aleatoires", *Studia Math.*, 19, pp. 1–25, 1960.

[9] V. Kvaratskhelia, V. Tarieladze and N. Vakhania, "Characterization of γ-Subgaussian Random Elements in a Banach Space", *Journal of Mathematical Sciences*, vol. 216, no. 4, pp. 564–568, 2016.

[10] A. Pietsch, "Absolute p-summierende abbildugen in normierten raumen", *Studia Math.*, vol. 28, pp. 333–353, 1967.

[11] W. Slowikowski, "Absolutely 2-summing mappings from and to Hilbert spaces and a Sudakov Theorem", *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, vol. 17, pp. 381–386, 1969.

[12] M. Talagrand, "Regularity of gaussian processes", *Acta Math.*, vol. 159, no. 1-2, pp. 99–149, 1987.

[13] N. Vakhania, "On subgaussian random vectors in normed spaces", *Bull. Georgian Acad. Sci.*, vol. 163, no. 1, pp. 8–11, 2001.

[14] N. N. Vakhania, V. V. Kvaratskhelia and V. I. Tarieladze, "Weakly sub-Gaussian random elements in Banach spaces", *Ukrainlan Math. J.*, vol. 57, no.9, 1387–1412, 2005.

[15] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan, *Probability distributions on Banach spaces.* Dordrecht: Reidel, 1987.

Բանախի տարածություններում ենթագաուսականության մասին

Ջորջ Դ. Գիորգոբիանի, Վախթանգ Վ. Կվարացխելիա և Վաժա Ի. Տարիելաձե

Վրաստանի տեխնիկական համալսարանի հաշվողական մաթեմատիկայի Մուսխելիշվիլու ինստիտուտ
Թբիլիսի, Վրաստան
e-mail: giorgobiani.g@gtu.ge, v.kvaratshelia@gtu.ge, v.tarieladze@gtu.ge

Ամփոփում

Մենք ցույց ենք տալիս, որ եթե X-ը Բանախի տարածություն է, և թույլ ենթագաուսյան պատահական տարրը X-ում առաջացնում է 2-ամփոփիչ օպերատորը, ապա այն T^* ենթագաուսյան է՝ պայմանով, որ X-ը ռեֆլեքսիվ 2 տիպի տարածություն է: Օգտագործելով այս արդյունքը, մենք ստանում ենք թույլ ենթագաուսյան պատահական տարրերի բնութագրում Հիլբերտյան տարածության մեջ, որոնք T^* ենթագաուսյան են:

Բանալի բառեր՝ ենթագաուսյան պատահական մեծություն ն, Գաուսյան պատահական մեծություն, թույլ ենթագաուսյան պատահական տարր, T^* ենթագաուսյան պատահական տարր, Բանախի տարածություն, Հիլբերտի տարածություն:

О субгауссовости в банаховых пространствах

Георгий Д. Гиоргобиани, Вахтанг В. Кварацхелия и Важа И. Тариеладзе

Институт вычислительной математики Мусхелишвили Грузинского технического университета,
Тбилиси, Грузия
e-mail: giorgobiani.g@gtu.ge, v.kvaratshelia@gtu.ge, v.tarieladze@gtu.ge

Аннотация

Мы показываем, что если X - банахово пространство и слабо субгауссовский случайный элемент в X индуцирует оператор 2-суммирования, то оно T^*-субгауссово при условии, что X - рефлексивное пространство типа 2. Используя этот результат, мы получаем характеристику слабо субгауссовских случайных элементов в гильбертовом пространстве, которые являются T^* субгауссовыми.

Ключевые слова: субгауссовская случайная величина, гауссовская случайная величина, слабо субгауссовский случайный элемент, T^* субгауссовский случайный элемент, банахово пространство, гильбертово пространство.