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On Testing of Multiple Hypotheses of Continuous Probability Distributions Arranged into Two Groups

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Abstract

The optimal Neyman-Pearson procedure of detection is investigated for models characterized by four continuous probability distributions arranged into two groups considered as hypotheses. It is worthy to note that the case of three discrete probability distributions arranged in two groups was studied by Haroutunian and Yesayan in [1]. The Neyman-Pearson theorem holds immense importance when it comes to solving problems that demand decision making or conclusions to a higher accuracy.

Keywords: Neyman-Pearson procedure, Continuous probability distribution, Probability density function, Hypothesis testing, Error probabilities, Reliabilities.

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1. Introduction

The Neyman-Pearson theorem states that the likelihood ratio test is the most powerful test for a given significance level (or size) in the context of simple binary hypothesis testing (null hypothesis against alternative hypothesis) problems. It provides a theoretical basis for determining the critical region or decision rule that maximizes the probability of correctly detecting a true effect while maintaining a fixed level of Type I error.

Statistical power represents the ability of a hypothesis test to detect a true effect or difference when it exists in the population. The theorem emphasizes the importance of optimizing this power while controlling the risk of both Type I and Type II errors. Type I error, also known as a false positive, occurs when we reject the null hypothesis (assuming an effect or difference exists) when it is actually true. Type II error, on the other hand, refers to a false negative, where we fail to reject the null hypothesis (assuming no effect or difference) when an effect or difference truly exists. The Neyman-Pearson theorem allows us to strike a balance between these errors by maximizing power while setting a predetermined significance level (the probability of Type I error).

In [2]-[4], Cox formulated several divers examples of problems for two families of hypotheses testing and developed a general modification of the Neyman-Pearson maximum-likelihood ratio procedures for the solution of such problems for the parameters of known continuous probability distributions (CPDs). In [1], Haroutunian and Yesayan studied the problems concerning the Neyman-Pearson criterion where discrete probability distributions are arranged in many groups and where the error probabilities decrease exponentially as 2^{-NE}, when the number of observations N (size of sample) tends to infinity. In [5], Tusnády studied the hypotheses testing problem of two CPDs, where error probabilities also exponentially approach zero. The optimal hypotheses testing problems, when error probabilities exponentially approach zero were also studied in [6] and in [7]-[9]. In [8], Haroutunian, Hakobyan and Hormosi-nejad studied on two-stage optimal testing of multiple hypotheses for the pair of families of discrete distributions. In [9], Yesayan and Gevorgyan solved the problem of many CPDs by means of two-stage asymptotically optimal testing of multiple hypotheses based on Tusnády's result.

The hypotheses testing problems for two hypotheses were described in detail by Borovkov [10], Levy [11], van Trees [12], Csiszár and Longo [13], Csiszár and Shields [14], Longo and Sgarro [15]. The Neyman-Pearson criterion of multiple hypotheses testing for discrete random variable was explored in [16].

This paper is devoted to the generalization of the Neyman-Pearson criterion for composite hypotheses testing problem of CPDs. The result is based on the method proposed by Thomas and Cover [17] in the paragraph of information theory and statistics.

2. Problem Presentation and Solution

Let $\mathcal{P}(\mathcal{X})$ be the space of all CPDs. Let X be a continuous random variable (CRV) with one of 4 possible CPDs given by probability density functions (PDFs) $f_m,\ m=\overline{1,4}$. Let $\mathbf{x}=(x_1,x_2,...,x_N)$, $x_n\in\mathcal{X},\ n=\overline{1,N}$, be a vector of results of N independent observations of the RV X, then the PDF will be $f_m^N(\mathbf{x})=\prod\limits_{n=1}^{N} f_m(x_n)$.

For a CRV X, four PDFs $f_1,\ f_2,\ f_3,\ f_4$ are given, called the hypotheses. A statistical hypothesis H is a conjecture about the distribution of population X.

The statistician should make a decision about CPD of CRV. In this paper, we consider this problem in two stage. These PDFs are divided into two groups (hypotheses) such that the first hypothesis H_1 is the group of $k=1,2,3$ PDFs and the second hypothesis is the group of $4-k$ PDFs. Let us consider the partition when $k=2$ and the hypotheses are as follows:

$$H_1:\{f_1,\ f_2\},\ \ H_2:\{f_3,\ f_4\}. \tag{1}$$

In the first stage the statistician must accept or reject the first hypothesis on the base of sample $\mathbf{x}$. If the first hypothesis is not rejected the statistician can detect which PDF (f_1 or f_2) corresponds to CRV. So, if it is rejected the second stage detection will be between f_3 and f_4.

Taking decisions about the hypotheses statistician can commit the errors.

The probability $\alpha_{l|l}^N$ is to accept a hypothesis different from the true hypothesis H_l, $l=1,2$.

We will show that the proposed Thomas and Cover's proof of Neyman-Pearson theorem for discrete probability distributions will also work for this case.

We will use these notations for the *maximum-likelihood* ratio procedure, so we will take the *maximum* of a pair of PDFs: $g_1^N(\mathbf{x})=\max({f_1}^N(\mathbf{x});{f_2}^N(\mathbf{x}))$, $g_2^N(\mathbf{x})=\max({f_3}^N(\mathbf{x});{f_4}^N(\mathbf{x}))$.

Theorem 1. *For the threshold $t \geq 0$, consider the test Ψ_N^* defined by region of acceptance $\mathcal{A}^{N*}$ for hypothesis H_1:*

$$\mathcal{A}^{N*} = \left\{ \mathbf{x} : \frac{g_1^N(\mathbf{x})}{{g_2}^N(\mathbf{x})} > t \right\},$$

and acceptance region $\overline{\mathcal{A}^{N*}}$ for H_2.

So, by these definitions, the corresponding error probabilities (mentioned also in the introduction) will be

$$\alpha_{1|1}^{N*}(t) = \alpha_{2|1}^{N*}(t) = g_1^N(\overline{\mathcal{A}^{N*}}) = \int_{\mathbf{x} \in \overline{\mathcal{A}^{N*}}} g_1^N(\mathbf{x}) d(\mathbf{x}),$$

$$\alpha_{2|2}^{N*}(t) = \alpha_{1|2}^{N*}(t) = g_2^N(\mathcal{A}^{N*}) = \int_{\mathbf{x} \in \mathcal{A}^{N*}} g_2^N(\mathbf{x}) d(\mathbf{x}).$$

Let $\mathcal{A}^N \subset \mathcal{X}^N$ be the decision region for H_1 of another test Ψ_N with error probabilities $\alpha_{1|1}^N$ and $\alpha_{2|2}^N$. If $\alpha_{1|1}^N \leq \alpha_{1|1}^{N*}$, then $\alpha_{2|2}^N \geq \alpha_{2|2}^{N*}$.

Proof. Let $\Psi_{\mathcal{A}^{N*}}$ and $\Psi_{\mathcal{A}^N}$ be indicator functions of regions. The indicator function is 1, if the sample belongs to the corresponding region, and 0, otherwise. It is obvious that for all $\mathbf{x} \in \mathcal{X}^N$,

$$(\Psi_{\mathcal{A}^{N*}}(\mathbf{x}) - \Psi_{\mathcal{A}^N}(\mathbf{x}))(g_1^N(\mathbf{x}) - t{g_2}^N(\mathbf{x})) \geq 0.$$

Then

$$\int_{\mathbf{x} \in \mathcal{X}^N} (\Psi_{\mathcal{A}^{N*}}(\mathbf{x}) g_1^N(\mathbf{x}) - t\Psi_{\mathcal{A}^{N*}}(\mathbf{x}){g_2}^N(\mathbf{x}) - \Psi_{\mathcal{A}^N}(\mathbf{x}) g_1^N(\mathbf{x}) + t\Psi_{\mathcal{A}^N}(\mathbf{x}){g_2}^N(\mathbf{x}))d(\mathbf{x})$$

$$= \int_{\mathbf{x} \in \mathcal{A}^{N*}} (g_1^N(\mathbf{x}) - t g_2^N(\mathbf{x}))d(\mathbf{x}) - \int_{\mathbf{x} \in \mathcal{A}^N} (g_1^N(\mathbf{x}) - t g_2^N(\mathbf{x}))d(\mathbf{x})$$

$$= (1 - \alpha_{1|1}^*) - t\alpha_{2|2}^* - (1 - \alpha_{1|1}) + t\alpha_{2|2} = (\alpha_{1|1} - \alpha_{1|1}^*) + t(\alpha_{2|2} - \alpha_{2|2}^*) \geq 0.$$

So, from $\alpha_{1|1} \leq \alpha_{1|1}^*$ it follows that $\alpha_{2|2} \geq \alpha_{2|2}^*$.

3. Conclusion

This paper discussed a suitable strategy of hypotheses testing for models with 4 known CPDs grouped in 2 clusters, considered as hypotheses. This problem can be generalized for $M > 4$ hypotheses, which can be grouped into 2 clusters in various combinations, i.e., the first hypothesis will be composed by $K = 1, 2, ...M - 1$ PDFs and the second by $M - K$ PDFs. The solving method will be the same, but it is obvious that the result of each combination will be different.

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Երկու խմբում դասավորված անընդհատ հավանականային բաշխումների վերաբերյալ բազմակի վարկածների ստուգում

Արամ Օ. Եսայան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ, Երևան, Հայաստան
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Ամփոփում

Նեյման-Պիրսոնի ստուգման օպտիմալ ընթացակարգը հետազոտվում է այն մոդելների համար, որոնք բնութագրվում են չորս անընդհատ հավանականային բաշխումներով, որոնք բաժանված են որպես վարկածներ դիտակվող երկու խմբի մեջ: Հարկ է նշել, որ երկու խմբի մեջ բաժանված երեք դիսկրետ հավանականային բաշխումների դեպքը հետազոտվել է Հարությունյանի և Եսայանի կողմից [1]:

Նեյման-Պիրսոնի թեորեմը մեծ նշանակություն ունի, երբ խոսքը վերաբերում է այնպիսի խնդիրների լուծմանը, որոնք պահանջում են որոշումներ կայացնել կամ ավելի բարձր ճշգրտությամբ եզրակացություններ անել:

Բանալի բառեր՝ Նեյման-Պիրսոնի տեստ, անընդհատ բաշխում, խտության ֆունկցիա,վարկածների ստուգում, սխալի հավանականություն, հուսալիություն:

О проверке многих гипотез непрерывных распределений вероятностей расположенных в двух группах

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Аннотация

Исследуется оптимальная процедура тестирования Неймана-Пирсона для моделей, характеризующихся четырьмя непрерывными распределениями вероятностей, разбитыми на две группы, рассматриваемые как гипотезы. Примечательно, что случай трех дискретных распределений вероятностей, расположенных в двух группах, был изучен Арутюняном и Есаяном [1].

Теорема Неймана-Пирсона имеет огромное значение, когда речь идет о решении задач, требующих принятия решений или выводов с более высокой точностью.

Ключевые слова: Критерий Неймана-Пирсона, непрерывное распределение, функция плотности, проверка гипотез, вероятность ошибки, надежность.