Mathematical Problems of Computer Science 60, 9–16, 2023.
doi: 10.51408/1963-0104

UDC 519.6

Image Reconstruction Using the *sinc* Kernel Function

Suren B. Alaverdyan

Institute for Informatics and Automation Problems of NAS RA,Yerevan, Armenia
e-mail: souren@iiap.sci.am

Abstract

This study is devoted to address the challenge of solving ill-posed integral equations for image restoration. The integral equation is widely recognized as an ill-posed problem [1]. Our study demonstrates that utilizing a two- dimensional function as a kernel function in the integral equation leads to stable solutions, by establishing a consistent dependence between the solutions and the input data (images).

We were able to obtain solutions for the integral equation without employing a regularization process, which significantly reduces the duration of the calculation process.
Keywords: Integral equation, Correct, Kernel function.
Article info: Received 16 March 2023; sent for review 17 April 2023; accepted 18 May 2023.

1. Introduction

There are many publications on the subject of image restoration, and ongoing research indicates its continued importance in the field.

Numerous methods have been developed for image recovery, including some well-known examples such as Wiener's method of filtration [2], Tikhonov's regularization method for solving numerical solutions to ill-posed double Fredholm integral equations of the first kind [1, 3], and a family of methods that utilize the bundle theorem's result. They are known as blind deconvolution. Additionally, some methods rely on spectral analysisto restore the image by altering the spectrum values corresponding to low and high frequencies.

It is worth noting that in solving the image restoration problem, additional challenges may arise, such as estimating the degree of image distortion, determining the radius of the scattering function, and finding the image sharpness coefficient or estimation.

The precision coefficient can be used as a characteristic or be one of the characteristics of the completion of the iteration process.

Since images can have both low-frequency and high-frequency noises simultaneously, the filtering problem becomes quite delicate. Implementing one filtering method can adversely affect another one, and vice versa. To address this issue, spectral analysis can be used to simultaneously alternate the spectrum values in the low-frequency and high-frequency ranges.

Considering the importance of V. Kotelnikov's signal recovery theorem, which uses orthogonal basis functions $\eta\ y_n = sinc(x-n),\ n \in Z$, however, when using the two- dimensional $sinc(x-n, y-\eta) = sinc(x-n)sinc(y-\eta)$ function as a kernel function in the integral equation for image recovery, a fundamental question arises whether this approach can make it a well-posed problem or not. Implementation of the algorithm in the time domain has yielded positive results making this work worthwhile.

2. Integral Equation for Image Recovery

The integral equation for image reconstruction is a two-dimensional Fredholm integral equation of the first type (the unknown function is contained within the subintegral expression) and is expressed as follows:

$$\int\limits_a^b \int\limits_c^d k(x-\xi, y-\eta) f(\xi,\eta) d\xi d\eta = g(x,y), \tag{1}$$

where $a \le x, \xi \le b$, $c \le y, \eta \le d$.

The function $k(x-\xi, y-\eta)$ is called a kernel of the integral equation and is also known as a point dispersion (or spread) function, and the integral equation has a free term: $g(x,y)$, which is a given function (representing approximate data, images, etc.), $f(\xi,\eta)$ is an unknown function. This equation characterizes a variety of other physical processes, such as tomography and chemical-mechanical smoothing [4].

It is assumed that the function $k(x,y)$ is a quadratic integrable function:

$$\int\limits_a^b \int\limits_c^d |k(x,y)|^2 dxdy < \infty.$$

Here are some examples of kernel functions:

- Gaussian kernel: $k(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$;
- Distance-based kernel: $k(x,y) = \frac{1}{\sqrt{x^2+y^2}}$;
- sinc kernel: $k(x,y) = sinc(x,y) = sinc(x)sind(y)$.

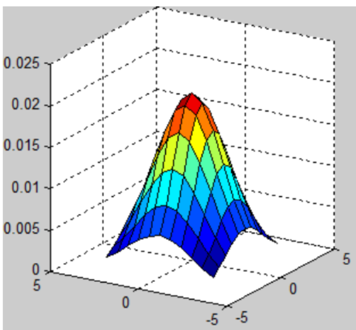

Fig. 1. Gaussian kernel.

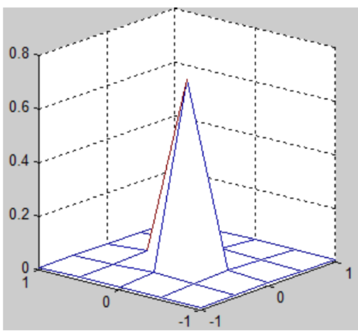

Fig. 2. Distance-based kernel.

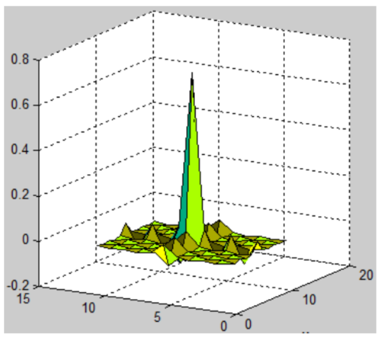

Fig. 3. sinc kernel.

3. $sinc(x)$ Function Properties

Let's discuss some properties that are significant in various fields of data processing. Most of these properties are used in the implementation of the algorithm.

$$\int_{-\infty}^{+\infty} sinc(x)dx = \pi.$$

The formula shows that the normalized $sinc(x)$ function takes the following form:

$$sinc(x) = \frac{sinc(\pi x)}{\pi x}.$$

The following relation is valid for integers:

$$sinc(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0: \end{cases} \tag{2}$$

The Fourier transform of $sinc(x)$ is called an orthogonal function, which has the following form:

$$\int_{-\infty}^{+\infty} sinc(x)e^{-2\pi i\omega x}dx = rect(\omega), \tag{3}$$

where the bounded rectangular function is expressed as follows:

$$rect(\omega) = \begin{cases} 1, & if \quad |\omega| \leq a, \\ 0, & if \quad |\omega| > a, \end{cases}$$

where $a = const$.

For practical applications, it is more convenient to represent the function $rect(\omega)$ in the following way:

$$rect(\omega) = \begin{cases} 1, & if \ \ 0 \leq \omega \leq 1, \\ 0, & out \ \ of \ \ range. \end{cases}$$

The frequency value of $\omega_0 = 0.5$ is commonly referred to as a center frequency. The frequency range of $\omega_0 < 0.5$ is considered as a low-pass-frequency range while $\omega_0 > 0.5$ is referred to as the high-frequency range. It is worth noting that at the value of $\omega_0 = 0.5$, the filtered function retains its original value.

We can show that the bounded delta function with the integral representation is reduced to the $sinc(x)$ function:

$$\delta(\omega) = \int_{-a}^{a} e^{-2i\omega t}dt = 2\int_{0}^{a} cos(\omega a)dt = \frac{2asin(\omega a)}{\omega a} = 2asin(\omega a). \tag{4}$$

The delta function has a filtering property. Let us now examine the graphs of the $sinc(x)$ function and its spectrum (see Fig. 4).

Signals with constant spectral density are called white noise, which contains the entire range of frequencies: $(0, \infty)$. By definition, the function $rect(\omega)$ is the spectral density of low- frequency $\omega < 5$ limited white noise. From (3) and (4) it follows that the $sinc$ function is the covariance function of low-frequency smooth noise.

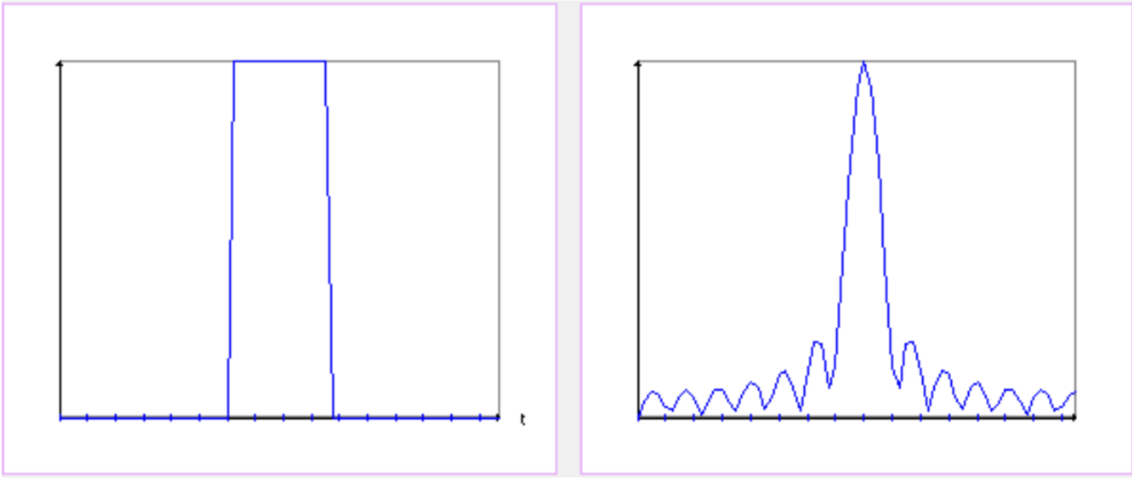

Fig. 4. Graphs of the $sinc(x)$ function and its amplitude spectrum.

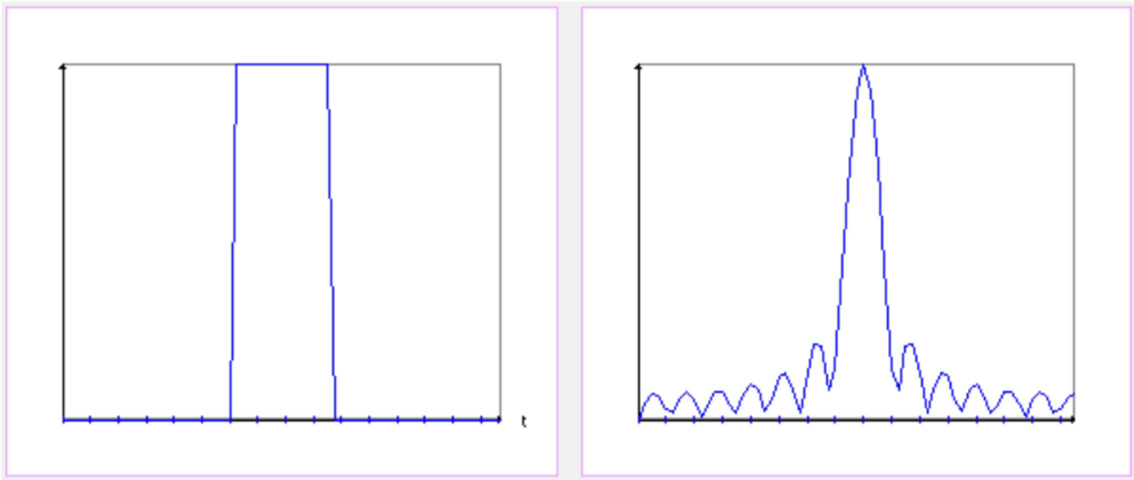

Fig. 5. Graphs of the $rect(x)$ function and its amplitude spectrum.

Formula (2) shows that for all values of n, the vectors $y_n = sinc(x-n),\ n \in Z$ form an orthonormal basis, which is used in signal recovery (V. Kotelnikov's theorem).

According to the theorem (V. Kotelnikov), any function $f(t)$, consisting of frequencies from 0 tof_c, can be transmitted continuously with any precision, using uniformly spaced samples taken at intervals of $1/(2f_c)$ seconds.

Any function $f(t)$ containing frequencies between 0 and f_c can be transmitted continuously with any level of precision using samples taken at intervals of $1/(2f_c)$ seconds.

$$x(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) sinc\left[\frac{\pi}{\Delta}(t-k\Delta)\right], \quad 0 < \Delta \leq \frac{1}{2f_c}.$$

For a signal to be accurately restored, it needs to be broadcast at a frequency of more than twice its maximum frequency. For instance, audio signals are commonly broadcast at a frequency of 44,000 hertz, given that the highest frequency audible to humans is 20,000 hertz. Additionally, Formula (4) shows the equivalence of the **delta** function of the $sinc(x)$ function. In physics, problems involving the **delta** function are typically handled by using the $sinc(x)$ function within a small range during calculations.

4. The *sinc* Function and Image Reconstruction in the Time Domain

An image is a projection of reflected electromagnetic waves onto a receiver. The main characteristics of an image, such as its resolution (the number of points per unit area), illumination, color, and contrast. These characteristics are significantly different from those characteristics of the signal that created it, such as its frequency, amplitude, and phase. However, there is a commonality between them: both are the result of wave processes.

In images, frequency and phase have a hidden nature. It becomes an object of study after the determination of its spectrum. The mentioned generality allows us to assume that the image is also a signal and represent it as a linear combination of the two-dimensional function $sinc(x-\xi)sinc(y-\eta)$ and an unknown function.

In this case, the integral equation (1) will take the following form:

$$\int_a^b \int_c^d sinc(x-\xi, y-\eta) f(\xi,\eta) d\xi d\eta = g(x,y). \tag{5}$$

To implement *sinc* filtering in the time domain, we can present (5) in a discrete form, where the $sinc(x)$ function will appear in a normalized form:

$$sinc(x-\xi, y-\eta) f(\xi,\eta) = \frac{sin(2\pi\omega(x-\xi))}{2\pi\omega(x-\xi)} \times \frac{sin(2\pi\omega(y-\eta))}{2\pi\omega(y-\eta)}. \tag{6}$$

Let m, n be the number of splits in $[a,b]$ and $[c,d]$ intervals:

$$dy = \frac{b-a}{m},\ dx = \frac{d-c}{n}.$$

We can represent the functions $f(\xi,\eta)$, $g(x,y)$, $sinc(x)sinc(y)$ as matrices F, G and S, respectively.

The G matrix has dimensions $m \times n$, while the matrix corresponding to the $sinc(x)sinc(y)$ kernel function is a square matrix, the size of which is optional: $k = 2p+1$. This number is chosen as an odd value to ensure the symmetry of transformations and prevent data skewing. Typically, algorithms are implemented for values of k=3.5, as larger values dramatically increase computation time.

To represent the unknown matrix F from (5) in the time domain, we use the known matrix G and a discrete package (such as convolution) of the kernel function (6):

$$F = G \cdot S,$$

where $\cdot$ is the convolution operation symbol.

The dimensions of the F matrix are $(m+2p) \times (n+2p)$. The formula for the discrete package looks as follows:

$$f_{i,j} = \sum_{u=i-p}^{i+p} \sum_{v=j-p}^{j+p} g_{i,j} s_{u-i+p, v-j+p}, \quad i = p, p+1, ..., m-p; \ \ j = p+1, ..., n-p. \tag{7}$$

It is important to note that while implementing the algorithm, the dimensions of the F matrix may not be changed, but the indices u and v should be controlled to ensure that they remain within the ranges $[0,m)$ and $[0,n)$.

Here is an example of the result of filtering an image containing Gaussian noise using (7) (see Fig. 6).

a b c

Fig. 6. a- original image, b- containing noise, $\sigma = 2$, $\rho = 9$, c- recovered image.

5. *sinc* Function and Image Restoration in the Spectral Domain

Image restoration in the spectral domain is performed by finding a solution to the linear integral Equation (5). To find the unknown function, the two-dimensional Fourier transform and the bundle theorem (convolution theorem) are used.

The set of functions $f(x)$ and $g(x)$ of one variable is the following integral:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt. \tag{8}$$

To understand the meaning of the package formula, let's consider a simple example that deals with the product of two polynomials. Consider two polynomials $P_1(x) = 2x - 1$; $P_2(x) = x^2 - 3x + 1$. We are required to find the product polynomial of these polynomials.

Let's write the coefficients of these polynomials in the following form, starting with the coefficients of the polynomial $P_1(x)$ in reverse order: $(-1, 2)$.

$$\begin{array}{llllllll} 1) & P_2(x): & 1,\ -3,\ 1 & 2) & 1,\ -3,\ 1 & 3) & 1,\ -3,\ 1 & 4) \quad 1,\ -3,\ 1 \\ & P_1(x): & \qquad\quad -1,\ 2 & & \quad -1,\ 2 & & \qquad -1,\ 2 & \qquad\quad -1,\ 2 \end{array}$$

It can be observed that at each step, the coefficients of the polynomial $P_1(x)$ are shifted one step to the left. The coefficients of the first polynomial are multiplied by the coefficients in front of them and added up. As a result we get the following:

$$1)\ -1 \cdot 1 = -1; \quad 2)\ -1 \cdot (-3) + 1 * 2 = 5; \quad 3)\ \ -1 \cdot 1 + 2 * (-3) = -7; \quad 4)\ 2 \cdot 1 = 2.$$

We obtain the coefficients of the product polynomial in reverse order. Correcting them we will get the following numbers: 2, -7, 5, -1.

$$P_1(x)P_2(x) = 2x^3 - 7x^2 + 5x - 1.$$

This calculation helps to characterize the significance of the bounded integral in Formula (8). Using the Fourier transform, the product of two functions can be computed efficiently. If $\hat{F}$ and $\hat{G}$ are transforms of the of functions f and g, respectively, then the product is calculated by the following formula:

$$(f * g)(x) = \Phi^{-1}(\hat{F} \cdot \hat{G}),$$

where $(\cdot)\hat{F} \cdot \hat{G}$ is the product of the corresponding elements of the vectors, and Φ^{-1} is the inverse Fourier transform.

According to the bundle theorem, the right-hand side of Equation (5) is the two-dimensional package of functions $sinc(x - \xi, y - \eta)$ and $f(\xi, \eta)$, so it can be represented in the following form:

$$\Phi_2(f(\xi, \eta) \cdot sinc(x, y)) = \Phi_2(g(x, y)). \tag{9}$$

Φ_2 and Φ_2^{-1} are direct and inverse two-dimensional Fourier transforms.

Let us denote the Fourier transforms of the functions $f(\xi, \eta)$, $sinc(x, y)$, $g(x, y)$ as $\hat{F}$, $\hat{S}$ and $\hat{G}$, respectively. Since we are dealing with a linear equation and linear transformations, we can obtain the following expression from Formula (9):

$$\hat{F} \cdot \hat{S} = \hat{G}, \quad or \quad \hat{F} = \frac{\hat{G}}{\hat{S}},$$

where we find the unknown function:

$$f(x, y) = \Phi_2^{-1}\left(\frac{\hat{G}}{\hat{S}}\right).$$

Since image restoration is an iterative process, an estimate of the image quality is needed to determine when the process should stop. This estimate is typically based on the normalized image gradient and Laplacian, which characterize the contours of objects in the image.

Given an image A having $m \times n$ patches. The coefficient of sharpness is determined by the following formula:

$$s = \frac{1}{2mna_{max}}\left(\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{\partial^2 a_{i,j}}{\partial x^2} + \frac{\partial^2 a_{i,j}}{\partial y^2}\right)^2} + \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{\partial a_{i,j}}{\partial x} + \frac{\partial a_{i,j}}{\partial y}\right)^2}\right).$$

In the given example shown in Fig. 6, for the input image 6-a, which doesn't contain any noise, the image sharpness coefficient value is assumed to be $s = 0.656817$. Then, low-frequency Gaussian noise with *radiuse* $\sigma = 2$, $r = 9$ is then added to the image 6-b, and the sharpness coefficient is calculated again, resulting in $s = 0.254214$. After filtering in the time domain, the sharpness coefficient is improved to $s = 0.583647$, and after filtering in the spectral domain, the sharpness coefficient is further improved to $s = 0.570277$.

6. Conclusion

The paper presents a method for finding solutions to the integral equation for image restoration, which bypasses the regularization process of an incorrectly set integral equation. This is achieved by selecting an appropriate dispersion function (kernel) of the integral equation of image restoration. The bounded two-dimensional $sinc(x, y)$ function was chosen as the kernel.

References

[1] Ж. Адамар, *Задача Коши для линейных уравнений с частными производными гиперболического типа*, М. Наука, 1978.

[2] Н. Ахмед и К. Рао, *Ортогональные преобразования при обработке цифровых сигналов*, Москва, Связь, 1980.

[3] А. Н. Тихонов, «О регуляризации некорректно поставленных задач», *ДАН СССР*, т. 153, н. 1, сс. 49 - 52, 1963.

[4] R. Ghulghazaryan, S. Alaverdyan and D. Piliposyan, “Accuarate pressure calculation method for CMP modeling using Fourier analysis”, *Reviesed Selected papers, Computer Science and Information Technologies (CSIT)*, doi: 10.1109/CSITechnol.2019.8895113, pp. 43-46, 2019.

Պատկերների վերականգնումը *sinc* միջուկի ֆունկցիայի միջոցով

Սուրեն Բ. Ալավերդյան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ, Երևան, Հայաստան
e-mail: souren@iiap.sci.am

Ամփոփում

Աշխատանքում ներկայացվել է պատկերների վերականգնման ինտեգրալ հավասարման լուծումներ գտնելու մեթոդ, որը շրջանցում է ոչ կոռեկտ դրված ինտեգրալ հավասարման ռեգուլյարիզացիայի պրոցեսը: Այս հանգամանքը պայմանավորված է պատկերների վերականգնման ինտեգրալ հավասարման ցրման ֆունկցիայի (կորիզ, kernel) ընտրությամբ: Որպես կորիզ ընտրվել է սահմանափակ երկչափ $sinc(x, y)$ ֆունկցիան:

Բանալի բառեր՝ Ինտեգրալ հավասարում, կոռեկտ, միջուկ, ֆունկցիա::

Реконструкция изображения с использованием функции ядра *sinc*

Сурен Б. Алавердян

Институт проблем информатики и автоматизации НАН РА, Ереван, Армения
e-mail: souren@iiap.sci.am

Аннотация

В работе представлен метод нахождения решения интегрального уравнения восстановления изображения, который позволяет обойти процесс регуляризации некорректно заданного интегрального уравнения. Это обстоятельство обусловлено выбором функции ядра интегрального уравнения восстановления изображения. В качестве ядра была выбрана ограниченная двумерная функция $sinc(x, y)$.

Ключевые слова: Интегральное уравнение, корректный, ядро, функция.