Mathematical Problems of Computer Science 49, 18–25, 2018.

Two Generalized Lower Bounds for the Circumference

Mossine S. Koulakzian and Zhora G. Nikoghosyan

Institute for Informatics and Automation Problems of NAS RA
e-mail: mossine@hotmail.com, zhora@ipia.sci.am

Abstract

In 2013, the second author obtained two lower bounds for the length of a longest cycle C in a graph G in terms of the length of a longest path (a longest cycle) in $G-C$ and the minimum degree of G (Zh.G. Nikoghosyan, "Advanced Lower Bounds for the Circumference", Graphs and Combinatorics 29, pp. 1531-1541, 2013). In this paper we present two analogous bounds based on the average of the first i smallest degrees in $G-C$ for appropriate i instead of the minimum degree.

Keywords: Circumference, Minimum degree, Degree sums.

1. Introduction

Let c be the circumference - the length of a longest cycle of a graph G and δ the minimum degree in G.

In this paper we present the following two results.

Theorem 1. *Let C be a longest cycle in a graph G, $\hat{p}$ the order of a longest path in $G-C$ and μ the average of the first $\hat{p}$ smallest degrees in $G-C$. Then*

$$c \geq (\hat{p}+1)(\mu-\hat{p}+1).$$

Theorem 2. *Let C be a longest cycle in a graph G, $\hat{c}$ the order of a longest cycle in $G-C$ and μ the average of the first $\hat{c}$ smallest degrees in $G-C$. Then*

$$c \geq (\hat{c}+1)(\mu-\hat{c}+1).$$

Observing that $\mu \geq \delta$ in Theorems 1 and 2, we obtain the original lower bounds [2] as immediate corollaries in terms of $\hat{p}$, $\hat{c}$ and δ.

Theorem A [2]. *Let C be a longest cycle in a graph G and $\hat{p}$ the order of a longest path in $G-C$. Then*

$$c \geq (\hat{p}+1)(\delta-\hat{p}+1).$$

Theorem B [2]. *Let C be a longest cycle in a graph G and $\hat{c}$ the order of a longest path in $G-C$. Then*

$$c \geq (\hat{c}+1)(\delta-\hat{c}+1).$$

2. Definitions

We use Bondy and Murty [1] for terminology and notation not defined here, and consider only finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$ or just V; the set of edges by $E(G)$ or just E. For a subgraph H of G we also use $G-H$ short for $G-V(H)$, and $|H|$ short for $|V(H)|$.

Paths and cycles in G can be considered as connected subgraphs of G, having a maximum degree 0,1 or 2. The length of a path P and of a cycle Q, denoted by $l(P)$ and $l(Q)$, is $|V(P)|-1$ and $|V(Q)|$, respectively. We denote $l(P)=-1$ and $l(Q)=0$ if and only if $V(P)=V(Q)=\emptyset$. A graph is said to be Hamiltonian if its longest cycle passes through all of its vertices. The vertices and edges in G can be interpreted as cycles of lengths 1 and 2, respectively.

An (x,y)-path is a path with end vertices x and y. Given an (x,y)-path L of G we denote by $\overrightarrow{L}$ the path L with an orientation from x to y. If $u,v\in V(L)$ then $u\overrightarrow{L}v$ denotes the consecutive vertices on $\overrightarrow{L}$ from u to v in the direction specified by $\overrightarrow{L}$. The same vertices, in reverse order, are given by $v\overleftarrow{L}u$. For $L=x\overrightarrow{L}y$ and $u\in V(L)$, let $u^+(\overrightarrow{L})$ (or just u^+) denote the successor of u $(u\neq y)$ on $\overrightarrow{L}$ and u^- denote its predecessor $(u\neq x)$. If $A\subseteq V(L)-y$ and $B\subseteq V(L)-x$, then we denote $A^+=\{v^+|v\in A\}$ and $B^-=\{v^-|v\in B\}$. A similar notation is used for the cycles. If Q is a cycle and $u\in V(Q)$, then $u\overrightarrow{Q}u=u$. For $v\in V$, put $N(v)=\{u\in V|uv\in E\}$, $d(v)=|N(v)|$ and $\delta=\min\{d(u)|u\in V\}$.

3. Special Definitions

For the remainder of this section, let a subgraph F of a graph G and a path (or a cycle) $\overrightarrow{M}$ in $G-F$ be fixed.

Definition 1. $(*i)$-minimality, $(*i)$-maximality.

We use the notions of $(*i)$-minimality and $(*i)$-maximaliy defined with respect to certain operations for $i=1,2,...,10$. They will be described in detail currently.

Definition 2. *MF-extension; $\overrightarrow{T}(u)$; $\dot{u}$; $\ddot{u}$.*

For each $u\in V(M)$, let $\overrightarrow{T}(u)=u\overrightarrow{T}(u)\ddot{u}$ be a path in G, having only u in common with $V(M)$. If $V(T(u))\cap V(T(v))=\emptyset$ and $V(T(u))\subseteq V(G-F)$ for all distinct vertices $u,v\in V(M)$, then the forest T, defined by $\{T(u)|u\in V(M)\}$, is said to be MF-extension. If $\ddot{u}\neq u$ for some $u\in V(M)$, then we use $\dot{u}$ to denote $u^+(\overrightarrow{T}(u))$.

Definition 3. *Φ_u; $\varphi(u)$; $\Psi(u)$; $\psi(u)$.*

Let T be an MF-extension. For each $u\in V(M)$, put

$$\Phi_u=N(\ddot{u})\cap V(T),\quad \varphi_u=|\Phi_u|,$$

$$\Psi_u=N(\ddot{u})\cap V(F),\quad \psi_u=|\Psi_u|.$$

Definition 4. *U_0; $\overline{U}_0$; U_1; U^*.*

Let T be an MF-extension. Put

$$U_0 = \{u \in V(M) | u = \ddot{u}\}; \quad \overline{U}_0 = V(M) - U_0,$$

$$U^* = \{u \in \overline{U}_0 | \Phi_u \subseteq V(T(u))\}; \quad U_1 = V(M) - (U_0 \cup U^*).$$

Definition 5. *Maximal MF-extension.*

An MF-extension T is said to be maximal if it is extremal with respect to the following operation: - if there exists an edge $\ddot{u}z$ such that $u \in V(M)$ and $z \notin V(T) \cup V(F)$, then replacing $T(u)$ by $uT(u)\ddot{u}z$, we obtain a new MF-extension T' with $|V(T')| > |V(T)|$.

Definition 6. *(U_0)-minimal and (U_0, U^*)-minimal MF-extensions.*

An MF-extension T is said to be (U_0)-minimal, if it is chosen such that U_0 is $(*6)$-minimal (see the proof of Theorem 1). A (U_0)-minimal MF-extension T is said to be (U_0, U^*)-minimal if it is chosen such that U^* is $(*10)$-minimal (see the proof of Theorem 2).

Definition 7. B_u; B_u^*; b_u; b_u^*.

Let T be an MF-extension and $u \in V(M)$. Put $B_u = \{v \in U_0 | v\ddot{u} \in E\}$ and $b_u = |B_u|$. By the definition, $B_u = \emptyset$ for each $u \in U_0$. Furthermore, for each $u \in U_0$, let $B_u^* = \{v \in \overline{U}_0 | u\dot{v} \in E\}$ and $|B_u^*| = b_u^*$.

4. Preliminaries

The proofs of the following lemmas can be find in [2].

Lemma 1. *Let C be a cycle in a graph G and P a path in $G - C$. Let $\overrightarrow{P}_0, ..., \overrightarrow{P}_p$ be pairwise disjoint paths in $G - C$ with $\overrightarrow{P}_i = v_i \overrightarrow{P}_i w_i$ $(i = 0, 1, ..., p)$, having only $v_0, ..., v_p$ in common with P. Then either there is a cycle in G longer than C or*

$$|C| \geq \sum_{i=0}^{p} |Z_i| + \left| \bigcup_{i=0}^{p} Z_i \right|,$$

where $Z_i = N(w_i) \cap V(C)$ $(i = 0, 1, ..., p)$.

Lemma 2. *Let F be a subgraph of a graph G and R a longest cycle in $G - F$ with a (U_0)-minimal RF-extension T. Then either there is a cycle longer than R or $l(R) \geq \varphi_u + b_u + 1$ for each $u \in U_1$.*

Lemma 3. *Let F be a subgraph of a graph G and P a path in $G - F$ with a (U_0)-minimal PF-extension T. Then either there is a path longer than P or $l(P) \geq \varphi_u + b_u$ for each $u \in U_1 \cup U^*$.*

5. Proofs

Proof of Theorem 1. *Let $Q = u_0 ... u_q$ be a path in $G - C$ with a (U_0)- minimal QC-extension T. Assume without loss of generality that C is $(*1 - *4)$-extremal, and Q is $(*7 - *9)$ -extremal. Since G is non-Hamiltonian, we have $q \geq 0$.*

Claim 1. If $u \in U_0$ and $v \in \overline{U}_0$, then $\Phi_u \cap V(T(v)) \subseteq \{v, \dot{v}\}$.
Proof. Suppose otherwise. Let $z \in V(T(v)) - \{v, \dot{v}\}$. Then, replacing $T(u)$ and $T(v)$ by $uz\overrightarrow{T}(v)\ddot{v}$ and $v\overrightarrow{T}(v)z^-$, respectively, we can form (denote this operation by $(*6)$) a new QC-extension, contradicting the (U_0)- minimality of T. □

Claim 2. If $u \in U_0$, then $\varphi_u \leq q + b^*_u$.
Proof. The proof follows immediately from Definitions 3, 7 and Claim 1. □

Claim 3. If $u \in \overline{U}_0$, then $\varphi_u \leq q - b_u$.
Proof. Using Lemma 3 with the fact that Q is $(*7 - *9)$-extremal, we obtain $q \geq \varphi_u + b_u$ for each $u \in \overline{U}_0$, and the result follows. □

Observing that

$$\sum_{u \in U_0} b^*_u = \sum_{u \in \overline{U}_0} b_u$$

(by the definition) and using Claims 2 and 3, we obtain

$$\sum_{i=0}^{q} \varphi_{u_i} \leq q(q+1) + \sum_{u \in U_0} b^*_u - \sum_{u \in \overline{U}_0} b_u = q(q+1).$$

Suppose first that $\varphi_{u_i} + \psi_{u_i} \neq d(\ddot{u}_i)$ for some $i \in \overline{0, q}$. Then there exists an edge $\ddot{u}z$ such that $z \notin V(T) \cup V(C)$. Adding $\ddot{u}z$ to T we obtain a new QC-extension, contradicting the maximality of T (Definition 5). Now let $\varphi_{u_i} + \psi_{u_i} = d(\ddot{u}_i)$ $(i = 0, ..., q)$. Then

$$\sum_{i=0}^{q} \psi_{u_i} = \sum_{i=0}^{q} d(\ddot{u}_i) - \sum_{i=0}^{q} \varphi_{u_i} \geq \sum_{i=0}^{q} d(\ddot{u}_i) - q(q+1).$$

It follows, in particular, that

$$\max_i \{\psi_{u_i}\} \geq \frac{1}{q+1} \sum_{i=0}^{q} \psi_{u_i} \geq \frac{1}{q+1} \sum_{i=0}^{q} d(\ddot{u}_i) - q.$$

By Lemma 1,

$$c \geq \sum_{i=0}^{q} \psi_{u_i} + \max_i \{\psi_{u_i}\}$$

$$\geq (q+2) \left(\frac{1}{q+1} \sum_{i=0}^{q} d(\ddot{u}_i) - q \right) \geq (q+2)(\mu_q - q). \qquad \blacksquare$$

Proof of Theorem 2. Let $H = u_1 ... u_k u_1$ be a cycle in $G - C$ with an (U_0, U^*)-minimal HC-extension T. Let H be $(*5)$-extremal. Put

$$U^*_1 = \left\{ u \in U^* | \varphi_u \leq \frac{h}{2} \right\}, \quad U^*_2 = \left\{ u \in U^* | \varphi_u \geq \frac{h+1}{2} \right\}.$$

Claim 1. If $u \in U_0$ and $v \in \overline{U}_0$, then $\Phi_u \cap V(T(v)) \subseteq \{v, \dot{v}\}$.
Proof. The proof is very similar to that of Claim 1 in Theorem 1. □

Claim 2. If $u \in U_0$, then $\varphi_u \le h - 1 + b_u^*$.
Proof. Immediate from Definitions 3, 7 and Claim 1. □

Claim 3. If $u \in U_1$, then $\varphi_u \le h - 1 - b_u$.
Proof. Since H is $(*5)$-extremal, by Lemma 2, $h \ge \varphi_u + b_u + 1$ for each $u \in U_1$, and the result follows. □

Claim 4. If $u \in U^*$, then $\varphi_u \le h - 1 - b_u + \varphi_u - \frac{h}{2}$.
Proof. Since H is $(*5)$-extremal, by the standard arguments, $h \ge 2(b_u + 1)$ for each $u \in U^*$, and the result follows immediately. □

Claim 5. If $u \in U_1$, then $\varphi_u \le h - 1 - b_u$.
Proof. Immediate from Claims 3 and 4. □

If $U_2^* = \emptyset$, then by Claims 2 and 5, $\sum_u \varphi_u \le h(h-1)$. But then, as in Theorem 1, $c \ge (h+1)(\lambda_1 - h + 1)$, where $\lambda_1 = \frac{1}{h}\sum_{i=1}^k d(\ddot{u}_i) \ge \mu_h$. Now let $U_2^* \ne \emptyset$. Choose $v \in U_2^*$ such that

$$\varphi_v = \max_{u \in U_2^*}\{\varphi_u\}. \tag{1}$$

Claim 6. If $u \in U_2^*$, then $\varphi_u \le h - 1 - b_u + \varphi_v - \frac{h}{2}$.
Proof. Immediate from (1) and Claim 4. □

Using Claims 2, 5, 6 and recalling that $\sum_{u \in U_0} b_u^* = \sum_{u \in \overline{U}_0} b_u$ and $|U_0| + |U_1 \cup U_1^*| + |U_2^*| = h$, we get

$$\sum_u \varphi_u = \sum_{u \in U_0} \varphi_u + \sum_{u \in U_1 \cup U_1^*} \varphi_u + \sum_{u \in U_2^*} \varphi_u \le h(h-1) + |U_2^*|\left(\varphi_v - \frac{h}{2}\right). \tag{2}$$

By Definition 3, $\Phi_v \subseteq V(T(v))$. Let $v_1, ..., v_t$ be the elements of Φ_v^+, occurring on $\overrightarrow{T}(v)$ in a consecutive order with $v_t = \ddot{v}$. Clearly $t = |\Phi_v| = \varphi_v$. Put

$$N(v_i) \cap V(T) = \Phi_i', \quad N(v_i) \cap V(C) = Z_i' \quad (i = 1, ..., t). \tag{3}$$

If $\Phi_i' \cap (V(T) - V(T(v))) \ne \emptyset$ for some $i \in \overline{1,t}$, then replacing $T(v)$ by

$$v\overrightarrow{T}(v)v_i^- \ddot{v}\overleftarrow{T}(v)v_i,$$

we form (denote this operation by $(*10)$) a new HC-extension, contradicting the minimality of $|U^*|$. So, we can assume $\Phi_i' \subseteq V(T(v))$ $(i = 1, ..., t)$. Assume w.l.o.g. that $\max_i |\Phi_i'| = |\Phi_t'| = \varphi_v$. So,

$$\max_i |\Phi_i'| = |\Phi_t'| = |\Phi_v| = \varphi_v = t. \tag{4}$$

Since $\psi_{u_i} = d(u_i) - \varphi_{u_i}$ $(i = 1, ..., h)$ and $|Z_i'| = d(v_i) - |\Phi_i'|$ $(i = 1, ..., t-1)$, we have

$$\sum_{i=1}^h \psi_{u_i} + \sum_{i=1}^{t-1} |Z_i'| = \sum_{i=1}^h (d(u_i) - \varphi_{u_i}) + \sum_{i=1}^{t-1}(d(v_i) - |\Phi_i'|)$$

$$= \sum_{i=1}^{h} d(u_i) + \sum_{i=1}^{t-1} d(v_i) - \sum_{i=1}^{h} \varphi_{u_i} - \sum_{i=1}^{t-1} |\Phi_i'|. \tag{5}$$

Put

$$\lambda_2 = \frac{1}{h+t-1} \left(\sum_{i=1}^{h} d(u_i) + \sum_{i=1}^{t-1} d(v_i) \right) \geq \lambda_1 \geq \mu_h.$$

Case 1. $|U_2^*| = 1$.
By (2), (4) and (5),

$$\sum_{i=1}^{h} \psi_{u_i} + \sum_{i=1}^{t-1} |Z_i'| \geq (h+t-1)\lambda_2 - h(h-1) - t + \frac{h}{2} - \sum_{i=1}^{t-1} t$$

$$= (h+t-1)\lambda_2 - h^2 - t^2 + \frac{3h}{2}.$$

It follows, in particular, that

$$\max_i \{\psi_{u_i}, |Z_i'|\} \geq \lambda_2 - \frac{h^2 + t^2 - \frac{3h}{2}}{h+t-1} \geq \lambda_2 - \frac{3h}{2} + 2.$$

If $\lambda_2 \leq h-1$, then clearly $c \geq (h+1)(\lambda_2 - h + 1)$. Let $\lambda_2 \geq h \geq t+1$. Applying Lemma 1 to $Q = \ddot{v}\overleftarrow{T}(v)v\overrightarrow{H}v^-$, we get

$$c \geq \sum_{i=1}^{h} \psi_{u_i} + \sum_{i=1}^{t-1} |Z_i'| + \max_i \{\psi_{u_i}, |Z_i'|\}$$

$$\geq (h+1)(\lambda_2 - h + 1) + (t-1)(\lambda_2 - t - 1) \geq (h+1)(\lambda_2 - h + 1).$$

Case 2. $|U_2^*| \geq 2$.
Choose $w \in U_2^* - v$ such that $\varphi_v \geq \varphi_w \geq \varphi_u$ for each $u \in U_2^* - \{v, w\}$. Define w_i, Z_i'', Φ_i'' $(i = 1, ..., r)$ for $T(w)$ in the same way as v_i, Z_i' and Φ_i' were defined for $T(v)$. As in (4), we can assume w.l.o.g. that $\max_i |\Phi_i''| = |\Phi_r''| = |\Phi_w| = \varphi_w = r$. Clearly, $t+r = \varphi_v + \varphi_w \geq h+1$. Then

$$\sum_{i=1}^{t} |Z_i'| + \sum_{i=1}^{r} |Z_i''| = \sum_{i=1}^{t} (d(v_i) - |\Phi_i'|) + \sum_{i=1}^{r} (d(w_i) - |\Phi_i''|)$$

$$= \sum_{i=1}^{t} d(v_i) + \sum_{i=1}^{r} d(w_i) - \sum_{i=1}^{t} |\Phi_i'| - \sum_{i=1}^{r} |\Phi_i''| \geq (t+r)\lambda_3 - t^2 - r^2,$$

where

$$\lambda_3 = \frac{1}{t+r} \left(\sum_{i=1}^{t} d(v_i) + \sum_{i=1}^{r} d(w_i) \right) \geq \mu_h.$$

In particular,

$$\max_i \{|Z_i'|, |Z_i''|\} \geq \lambda_3 - \frac{t^2 + r^2}{t+r}.$$

Applying Lemma 1 to $Q = \ddot{v}\overleftarrow{T}(v)v\overrightarrow{H}w\overrightarrow{T}(w)\ddot{w}$, we get

$$c \geq \sum_{i=1}^{t} |Z_i'| + \sum_{i=1}^{r} |Z_i''| + \max_i \{|Z_i'|, |Z_i''|\} \geq (t+r)\lambda_3 - t^2 - r^2 + \lambda_3 - \frac{t^2 + r^2}{t+r}$$

$$\geq (h+1)(\lambda_3 - h + 1) + \lambda_3(t + r - h) + h^2 - 1 - t^2 - r^2 - \frac{t^2 + r^2}{t + r}.$$

If $\lambda_3 \leq h - 1$, then clearly, $c \geq (h+1)(\lambda_3 - h + 1)$. Otherwise,

$$c \geq (h+1)(\lambda_3 - h + 1) + h(t + r) - 1 - t^2 - r^2 - \frac{t^2 + r^2}{t + r}$$

$$\geq (h+1)(\lambda_3 - h + 1) + (h-1)(t + r) - t^2 - r^2.$$

Observing that

$$h - 1 \geq \max\{t, r\} \geq \frac{t^2 + r^2}{t + r},$$

we obtain $c \geq (h+1)(\lambda_3 - h + 1)$. Thus, $c \geq (h+1)(\lambda - h + 1)$, where $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\} \geq \mu_h$. ■

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications. Macmillan, London and Elsevier, New York, 1976.

[2] Zh.G. Nikoghosyan, "Advanced lower bounds for the circumference", Graphs and Combinatorics 29, pp. 1531-1541, 2013.

Submitted 22.08.2017, accepted 24.12.2017.

Գրաֆի ամենաերկար ցիկլի երկարության երկու ընդհանրացված գնահատականներ

Մ. Քուլաքզյան և Ժ. Նիկողոսյան

Ամփոփում

2013-ին երկրորդ հեղինակը G գրաֆի ամենաերկար C ցիկլի երկարության համար ստացավ երկու ստորին գնահատականներ՝ արտահայտված $G - C$-ի ամենաերկար շղթայի երկարության (ամենաերկար ցիկլի երկարության) և G գրաֆի նվազագույն աստիճանի բնութագրիչներով (Zh.G. Nikoghosyan, Advanced Lower Bounds for the Circumference, Graphs and Combinatorics 29, pp. 1531-1541, 2013): Ներկա աշխատանքում ներկայացվում են երկու համանման գնահատականներ, որտեղ նվազագույն աստիճանի բնութագրիչը փոխարինված է $G - C$-ի գագաթների առաջին i ամենափոքր աստիճանների միջին թվաբանականով $G - C$-ով պայմանավորված որոշ i պարամետրի համար:

Две обобщенные нижние оценки для длины длиннейшего цикла графа

М. Кулакзян и Ж. Никогосян

Аннотация

В 2013 году второй автор получил две нижние оценки для длины длиннейшего цикла графа G выраженные через длину длиннейшей цепи (длиннейшего цикла) подграфа $G-C$ и минимальную степень графа G (Zh.G. Nikoghosyan, Advanced Lower Bounds for the Circumference, Graphs and Combinatorics 29, pp. 1531-1541, 2013)? В настоящей работе представляются две обобщенные аналогичные оценки, где вместо минимальной степени рассматривается средняя арифметическая степеней первых i наименьших степеней вершин подграфа $G-C$ для подходящего параметра i.