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Some Results on Palette Index of Cartesian Product Graphs

Khachik S. Smbatyan

Yerevan State University
e-mail: smbatyan1729@gmail.com

Abstract

Given a proper edge coloring α of a graph G, we define the palette $S_G(v,\alpha)$ of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident to v. The palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G. The windmill graph $Wd(n,k)$ is an undirected graph constructed for $k \geq 2$ and $n \geq 2$ by joining n copies of the complete graph K_k at a shared universal vertex. In this paper, we determine the bound on the palette index of Cartesian products of complete graphs and simple paths. We also consider the problem of determining the palette index of windmill graphs. In particular, we show that for any positive integers $n, k \geq 2$, $\check{s}(Wd(n,2k)) = n+1$.

Keywords: Edge coloring, Proper edge coloring, Palette, Palette index, Cartesian product, Windmill graph.
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1. Introduction

Throughout this paper, a graph G always means a finite undirected graph without loops, parallel edges, and it does not contain isolated vertices. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex v in G is denoted by $d_G(v)$, and the maximum degree of vertices in G by $\Delta(G)$. The terms and concepts that we do not define can be found in [1].

An *edge coloring* of a graph G is an assignment of colors to the edges of G: it is *proper* if adjacent edges receive distinct colors. The minimum number of colors required in a proper edge coloring of a graph G is called the chromatic index of G and denoted by $\chi'(G)$. By Vizings theorem [9], the chromatic index of G equals either $\Delta(G)$ or $\Delta(G)+1$. A graph with $\chi'(G) = \Delta(G)$ is called *Class 1*, while a graph with $\chi'(G) = \Delta(G)+1$ is called *Class 2*.

In this paper, we consider a chromatic parameter called the palette index of a simple graph G. A proper edge-coloring of a graph defines at each vertex $v \in V(G)$ the set of colors of its incident edges. That set is called the *palette of* v and denoted by $S_G(v,\alpha)$. The minimum number of palettes, taken over all possible proper edge colorings of a graph G, is called a *palette index* of a graph and denoted by $\check{s}(G)$ [2]. Proper edge colorings with the minimum number of distinct palettes were studied for the first time in 2014, by Horňák, Kalinowski, Meszka, and Woźniak [2]. They determined the palette index of complete graphs. Namely,

$$\check{s}(K_n) = \begin{cases} 1, & \text{if n} \equiv 0(\text{mod}2) \\ 3, & \text{if n} \equiv 3(\text{mod}4) \\ 4, & \text{if n} \equiv 1(\text{mod}4) \end{cases} \tag{1}$$

Moreover, they also showed that the palette index of a d-regular graph is 1 if and only if the graph is of $Class$ 1. If G is d-regular and of $Class$ 2, then Vizings edge coloring theorem [9] implies that $3 \le \check{s}(G) \le d+1$, and the case $\check{s}(G) = 2$ is not possible, as proved in [2]. There are few results about the palette index of non-regular graphs. Vizings edge coloring theorem also yields an upper bound on the palette index of a graph G with maximum degree Δ and without isolated vertices, mainly $\check{s}(G) \le 2^{\Delta+1} - 2$. In [6], Casselgren and Petrosyan provided an improvement and derived the following upper bound on the palette index of bipartite graphs:

$$\check{s}(G) \le \sum_{d \in D_{even}(G)} \binom{\left\lceil \frac{\Delta(G)}{2} \right\rceil}{\frac{d}{2}} + \sum_{d \in D_{odd}(G)} \binom{\left\lceil \frac{\Delta(G)}{2} \right\rceil}{\frac{d+1}{2}} (d+1) \tag{2}$$

where $D_{odd}(G)$ is the set of all odd degrees in G and $D_{even}(G)$ is the set of even degrees in G.

In [3], Bonvicini and Mazzuoccolo proved that if G is 4-regular and of Class 2, then $\check{s}(G) \in \{3, 4, 5\}$, and that all these values are, in fact, attainable. Although it is possible to determine the exact value of the palette index for some classes of graphs, in general, it is an NP-complete problem, because from [4] it is known that computing the chromatic index of a given graph is an NP-complete problem.

In this paper, we provide upper and lower bounds on the palette index of Cartesian products of some graphs. We will give the exact number of palettes of $Wd(n, 2k)$ windmill graphs, as well as the upper and lower bounds for $Wd(n, 2k+1)$.

2. Preliminaries

In this section, we introduce some terminology and notation. A matching in a graph G is a set of pairwise independent edges of G. A matching that saturates all the vertices of G is called a perfect matching. Next, we need some additional definitions.

Definition 1: *(Windmill graph). The windmill graph $Wd(n, k)$ is an undirected graph constructed for $k \ge 2$ and $n \ge 2$ by joining n copies of the complete graph K_k at a shared universal vertex.*

Definition 2: *(Cartesian product of graphs). Let G and H be two graphs. The Cartesian product $G \square H$ of graphs G and H is a graph such that*

- *the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$.*
- *two vertices (u, u_1) and (v, v_1) are adjacent in $G \square H$ if and only if either*

– *$u = v$ and u_1 is adjacent to v_1 in H, or*

– *$u_1 = v_1$ and u is adjacent to v in G.*

Before we move on, we recall that the Cartesian product graph $G\Box H$ decomposes into $|V(G)|$ copies of H and $|V(H)|$ copies of G. By the definition of Cartesian products of graphs, $G\Box H$ has two types of edges: those the vertices of which have the same first coordinate, and those the vertices of which have the same second coordinate. The edges joining vertices with a given value of the first coordinate form a copy of H, so the edges of the first type form nH ($|V(G)| = n$). Similarly, the edges of the second type form mG ($|V(H)| = m$), and the union is $G\Box H$.

Definition 3: *Given two graphs G and H, and a vertex $y \in V(H)$, the set $G^y = \{(x, y) \in V(G\Box H)|x \in V(G)\}$ is called a G-fiber in the Cartesian product of G and H. For $x \in V(G)$, the H-fiber is defined as ${}^xH = \{(x, y) \in V(G\Box H) \mid y \in V(H)\}$.*

G-fibers and H-fibers can be considered as induced subgraphs when appropriate. In [8], authors define the projection to G, which is the map $p_G : V(G\Box H) \to V(G)$ is defined by $p_G(x, y) = x$. Also we will need the projection to H; $p_H : V(G\Box H) \to V(H)$ is defined by $p_H(x, y) = y$.

In the proofs of our results, we also will follow some coloring ideas from [2]. Namely, we will use the coloring ideas described in the proofs of *Proposition 5*, which states that if $k \geq 0$, then $\check{s}(K_{4k+3}) = 3$, and *Theorem 7*, which shows that if $n = 4k + 5$, $k \neq 1$, then $\check{s}(K_n) = 4$.

3. Main Results

First, we will provide some results about the palette index of the Cartesian product of a cycle and simple path. Note that the palette index of $C_n\Box P_2$ is equal to 1. Clearly, the Cartesian product of those graphs is a *Class* 1 regular graph and as mentioned above the palette index of *Class* 1 regular graph is equal to 1.

Proposition 1: *If $n = 2k$ and $m > 2$, then $\check{s}(C_n\Box P_m) = 2$.*

Proof. First note that $C_n\Box P_m$ is not a regular graph, hence, $\check{s}(C_n\Box P_m) \geq 2$. Let construct a coloring that will induce 2 distinct palettes.

Case 1. m is even. Every $C_n - fiber$ can be properly colored alternately with colors a_1 and a_2. Because of the even length of cycles, we will get exactly one palette, denote it by $\{a_1, a_2\}$. Next, there are $n - pieces$ of $P_m - fibers$, and every $P_m - fiber$ can be properly colored alternately with colors a_3 and a_4. As a result, the palette of vertices with degree 3 is $\{a_1, a_2, a_3\}$, and the palette of vertices with degree 4 is $\{a_1, a_2, a_3, a_4\}$.

Case 2. m is odd. Suppose that $V(P_m) = \{v_1, v_2, ..., v_m\}$ and for any $i(1 \leq i \leq m - 1), v_iv_{i+1} \in E(P_m)$. Let $\alpha : E(C_n) \to \{a_1, a_2\}$ be a proper edge coloring of C_n. Since $C_n^{v_i}, 1 \leq i \leq m$ is isomorphic to C_n; hence, $C_n^{v_i}$ $(4 \leq i \leq m)$ can be properly colored with colors from the color-set $\{a_1, a_2\}$: $\forall (u, v_i), (u', v_i) \in V(C_n^{v_i})$ if $(u, v_i)(u', v_i) \in E(C_n\Box P_m)$, then we define a proper edge coloring γ as follows:

$$\gamma((u, v_i)(u', v_i)) = \alpha(p_G(u, v_i)p_G(u', v_i)) = \alpha(uu') = a,$$

where $a \in \{a_1, a_2\}$. Afterwards, the fibers $C_n^{v_1}$, $C_n^{v_2}$ and $C_n^{v_3}$ can be colored alternately with colors from the color-sets $\{a_1, a_2\}$, $\{a_1, a_4\}$ and $\{a_1, a_3\}$, respectively. Then we will color the edges joining $C_n^{v_1}$ to $C_n^{v_2}$ and $C_n^{v_2}$ to $C_n^{v_3}$ by the colors a_3 and a_2, respectively. Observe that the remaining uncolored edges of $P_m - fibers$ can be properly colored alternately with colors a_3 and a_4; the obtained coloring γ is a proper edge coloring of $C_n \square P_m$ with a minimum number of palettes. ■

Using the same ideas makes it easy to obtain a coloring for $C_{2n+1} \square P_{2m}$, inducing 2 distinct palettes. When the number of vertices of the cycle and the number of vertices of the path are odd, we have the following theorem.

Theorem 1: *If $n = 2k_1 + 1$ and $m = 2k_2 + 1$, $k_1, k_2 > 0$, then*

$$\check{s}(C_n \square P_m) = 4.$$

Proof. Suppose that $V(P_m) = \{v_1, v_2, ..., v_m\}$ and $d_{P_m}(v_1) = d_{P_m}(v_m) = 1$ and α is a coloring of $C_n \square P_m$ inducing $\check{s}(C_n \square P_m)$ distinct palettes. Let show that the value of the palette index is at least 4.

Case 1. $\check{s}(C_n \square P_m) = 1$. It follows that the graph is a regular graph, which is a contradiction.

Case 2. $\check{s}(C_n \square P_m) = 2$. Denote by P_1 and P_2 palettes induced by α. Clearly, $P_1 \cap P_2 \neq \emptyset$, therefore there is a color $a \in P_1 \cap P_2$ so that the edges colored with a form a perfect matching of the graph. However, $|V(C_n \square P_m)|$ is an odd number, which means that the graph cannot have a perfect matching, a contradiction.

Case 3. $\check{s}(C_n \square P_m) = 3$. Denote by P_1, P_2, and P_3 palettes induced by α. Suppose that $|P_1| = |P_2| = 3$ and $|P_3| = 4$. Clearly, there is no color belonging to all three palettes. Indeed, otherwise, that color would induce a perfect matching of $C_n \square P_m$, which is impossible. Assume that $(P_1 \cup P_2) \setminus P_3 \neq \emptyset$, then there is a color $a \in P_1 \cup P_2$ such that the edges colored with a form a perfect matching for C_n, which is impossible too, but this also means that the set $P_1 \cap P_2 \cap P_3$ cannot be empty, a contradiction.

Now, suppose that $|P_1| = |P_2| = 4$ and $|P_3| = 3$. Clearly there is a color $a \in P_1 \cap P_2$ and $a \notin P_3$. This implies that the edges colored with a form a perfect matching of $C_{n-2} \square P_m$, which is impossible. Hence, $\check{s}(C_n \square P_m) \geq 4$.

Next, we need to show the existence of a proper edge coloring α inducing four palettes. Assume that β is a proper edge coloring of C_n with colors from color-set $S = \{a_1, a_2, a_3\}$, inducing 3 distinct palettes. As we have already mentioned, $C_n^{v_i}, 1 \leq i \leq m-1$ can be properly colored with colors from the color-set S. Then for all $i (1 \leq i \leq m-1)$ the edge that joins $(u, v_i) \in V(C_n^{v_i})$ and $(u, v_{i+1}) \in V(C_n^{v_{i+1}})$ will be colored in one of the two ways, first if there is a color $a \in S$ that a does not belong to color-sets assigned to the incident edges of (u, v_i) and (u, v_{i+1}), then that edge will be colored with a. Otherwise it will be colored with a new color $a_4 \notin S$. Thereby we constructed coloring of the subgraph of $C_n \square P_m$, that is isomorphic to $C_n \square P_{m-1}$, inducing two palettes $\{a_1, a_2, a_3\}$ and $\{a_1, a_2, a_3, a_4\}$.

Note that the palette of the vertices of $C_n^{v_{m-1}}$ is $\{a_1, a_2, a_3\}$; hence, the colors assigned to the edges of $C_n^{v_{m-1}}$ divide that edge set into three disjoint sets: two sets X and Y, each having $\frac{n-1}{2}$ elements, and one one-element set, say $\{(u, v_{m-1})(u_1, v_{m-1})\}$. Without loss of generality, we may suppose that $(u, v_{m-1})(u_2, v_{m-1}) \in Y$ and X, Y are the sets of edges colored with a_1 and a_2, respectively. For all $(u', v_{m-1})(u'', v_{m-1}) \in X$ let do the following changes: $\alpha((u', v_{m-1})(u'', v_{m-1})) = a_4$, $\alpha((u', v_{m-1})(u', v_m)) = a_2$ and $\alpha((u'', v_{m-1})(u'', v_m)) = a_2$. Since (u, v_{m-1}) is the only vertex that the recent changes did not affect, $\alpha((u, v_{m-1})(u, v_m)) = a_4$. Finally, coloring the edge $\alpha((u, v_m)(u_1, v_m)) = a_5$ and the remaining uncolored edges alternately with colors a_2 and a_3 will induce two new palettes; hence, $\check{s}(C_n \square P_m) = 4$. ■

Next, we will examine the palette index of the Cartesian product of complete graphs and paths. Complete graph K_{2k} is of $Class$ 1 and $\check{s}(K_{2k}) = 1$, therefore $\check{s}(K_{2k} \square P_2) = 1$. On the other hand, the minimum coloring of K_{2k+1} induces $2k + 1$ distinct palettes. Indeed, each palette has $2k$ colors. This means that exactly one color is missing at each vertex. So we can use the minimum coloring of K_n for all $K_n - fibers$ and color the edges joining them with missing colors, hence, $\check{s}(K_{2k+1} \square P_2) = 1$.

Corollary 1. *If $n > 2$ and $m > 2$, then $\check{s}(K_{2n} \square P_m) = 2$.*

Proof. Construction of a proper edge coloring of $K_{2n} \square P_m$ is very similar to the steps that we have already described in Proposition 1, the single difference being that in this case we will color $K_n - fibers$ with the minimum coloring described above. ■

Theorem 2: *For any odd positive integers m and $k \geq 0$, we have*

$$\check{s}(K_{4k+3} \square P_m) = 4.$$

Proof. Let $V(P_m) = \{v_1, v_2, ..., v_m\}$ and $d_{P_m}(v_1) = d_{P_m}(v_m) = 1$. As we have already mentioned above there is a proper edge coloring with a minimum number of distinct palettes $\alpha : E(K_{4k+3}) \to S = \{a_1, a_2, a_3, ...a_{4k+3}\}$ inducing $4k+3$ different palettes. We will construct the coloring γ for $K_{4k+3} \square P_m$ as follows; $\forall i(1 \leq i \leq m - 1)$ and $\forall (u, v_i)(u', v_i) \in E(K_{4k+3}^{v_i})$ $\gamma((u, v_i)(u', v_i))$ will be set equal to $\alpha(uu')$. Note that for any $i(1 \leq i \leq m - 1)$, the vertices (u, v_i) and (u, v_{i+1}) are joined with the edges of $P_m - fibers$, and we have two possible cases for the coloring of these edges;

- if $S \setminus S_{K_{4k+3} \square P_m}((u, v_i)) = \{a\}$, then $\gamma((u, v_i)(u, v_{i+1})) = a$.
- if $S \setminus S_{K_{4k+3} \square P_m}((u, v_i)) = \emptyset$, then $\gamma((u, v_i)(u, v_{i+1})) = b$, $b \notin S$.

Note that the fiber $K_{4k+3}^{v_{m-1}}$ always has more than $k + 1$ edges colored with the same color. Assume that $M = \{(u_{i_1}, v_{m-1})(u_{i_2}, v_{m-1}), ..., (u_{i_{2k+1}}, v_{m-1})(u_{i_{2k+2}}, v_{m-1})\}$ is the set of edges colored with $a' \in S$. Now let recolor some edges. For any $j(1 \leq j \leq k + 1)$;

$$\gamma((u_{i_{2j-1}}, v_{m-1})(u_{i_{2j}}, v_{m-1})) = b,$$

$$\gamma((u_{i_s}, v_{m-1})(u_{i_s}, v_m)) = a', \ \forall s \in \{1, 2, ..., 2k + 2\},$$

$$\gamma((u_i, v_{m-1})(u_i, v_m)) = b, \ \forall u_i \in V(K_{4k+3}^{v_{m-1}}) \setminus \{u_{i_1}, u_{i_2}, ..., u_{i_{2k+2}}\}$$

To color the edges of $K_{4k+3}^{v_m}$, we will follow the coloring idea introduced in the proof of [2](Proposition 5). Using the color-set $S \cup \{b_1, b_2, ...b_{2k+1}\} \cup \{b\}$ and taking the vertex set $X = \{u_{i_1}, u_{i_2}, ..., u_{i_{2k}}\}$, $Y = V(K_n^{v_m}) \setminus (X \cup \{u_{i_{2k+1}}\})$ and one-element set $\{u_{i_{2k+1}}\}$ will let us obtain coloring that induces 2 new palettes. Clearly, we can make the palette of the vertices from the vertex set X equal to $\{a_1, a_2, a_3, ...a_{4k+3}\}$, causing new palettes only on the vertices from the vertex set Y and $\{u_{i_{2k+1}}\}$; hence $\check{s}(K_{4k+3} \square P_m) \leq 4$.

Now let us show that the palette index is at least 4. Suppose first that $\check{s}(K_{4k+3} \square P_m) = 3$, and let α be the corresponding coloring of $K_{4k+3} \square P_m$. Denote by P_1, P_2 and P_3 the palettes caused by α. Let $V_i = \{x \in V : S(x, \alpha) = P_i\}, i = 1, 2, 3$. First, there is no color belonging to all three palettes, otherwise this color would induce a perfect matching of $K_{4k+3} \square P_m$, which is impossible.

Case 1. $|P_1| = |P_2| = n$, $|P_3| = n + 1$. Note that $(P_1 \cup P_2) \setminus P_3 = \emptyset$; otherwise there is a color $a \in (P_1 \cup P_2) \setminus P_3$ then the edges colored with a form a perfect matching of K_n, which is impossible. It follows that $P_1 \cap P_2 \cap P_3 \neq \emptyset$, a contradiction.

Case 2. $|P_1| = |P_2| = n + 1$, $|P_3| = n$. Clearly, there is an edge $e \in E(K_{4k+3} \square P_m,)$ joining V_1 and V_2. Assume that $\alpha(e) \notin P_3$, then the edges colored with $\alpha(e)$ will form a perfect matching of $K_{4k+1} \square P_m$, which is a contradiction.

Suppose next that $\check{s}(K_{4k+3} \square P_m) = 2$, the intersection of the induced palettes similarly cannot be an empty set. Hence, $\check{s}(K_{4k+3} \square P_m) \neq 2$. ■

Also, note that constructed coloring will induce at most 5 palettes for $K_{4k+5} \square P_{2m+1}$, the single difference being that in this case we will color the fiber $K_{4k+5}^{v_m}$ using the coloring constructed in the proof of [2](Theorem 7).

Corollary 2. *If $k \geq 0$ and $m \geq 1$, then*

$$4 \leq \check{s}(K_{4k+5} \square P_{2m+1}) \leq 5$$

Next results are about the palette index of windmill graphs.

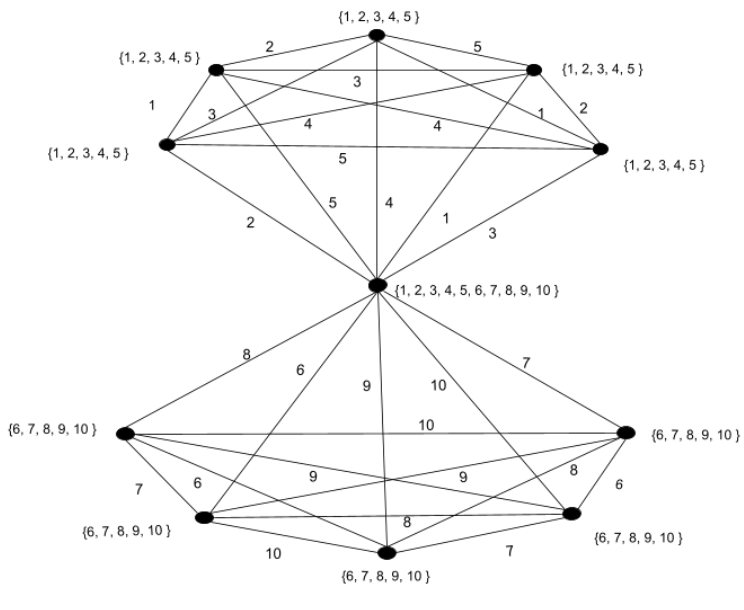

Fig. 1. $Wd(2, 6)$ graph coloring.

Proposition 2: *If $n, k \geq 2$, then*

$$\check{s}(Wd(n, k)) \geq n + 1.$$

Proof. Suppose that $\check{s}(Wd(n, k)) = m$ $(m < n + 1)$. There is a proper edge coloring of $Wd(n, k)$ inducing m distinct palettes P_i, $i = 1, 2, ..., m$. Let V_i be the set of all vertices of K_n with palette P_i and let $n_i = |V_i|$, $i = 1, 2, ..., m$. Without loss of generality, suppose that $|V_m| = n_m = 1$ is a one-element set, say $\{u\}$. Assume that u is the shared vertex of $Wd(n,k)$. Clearly, $\sum_{i=1}^{m} n_i = |V(Wd(n,k))| = n(k-1)+1$. This implies that $\exists i(1 \le i \le m)$ that $n_i \ge k$. Indeed, if $n_i < k(1 \le i \le m)$, then it follows;

$$\sum_{i=1}^{m} n_i = \sum_{i=1}^{m-1} n_i + 1 \le (m-1)(k-1) + 1 < n(k-1) + 1 = |V(Wd(n,k))|,$$

which is impossible. Thus, $\exists j$ such that $|V_j| = n_j > k$. For any vertex of V_j there is an edge joining it with shared vertex u, and the number of such edges is equal to n_j. On the other hand, $n_j > |P_j| = k-1$, which is a contradiction; therefore $\check{s}(Wd(n,k)) \ge n+1$. ■

At the same time the upper bound of the palette index of windmill graphs depends on the number of complete graphs.

Theorem 3: *For any positive integers $n, k \ge 2$, we have $\check{s}(Wd(n,2k)) = n+1$.*

Proof. We only need to show the existence of a coloring α inducing $n+1$ palette. Denote by u the shared vertex of $Wd(n,2k)$. Note that $Wd(n,2k) - u$ is a graph that consists of n components, and every component is a complete graph with $2k-1$ vertices. For every K_{2k-1} complete graph exists coloring inducing $2k-1$ palettes, and at each vertex, exactly one color is missing, which will be assigned to the edge joining that vertex and the shared vertex u. Clearly, this coloring will induce exactly one palette on every odd component, and as a result, we will construct coloring α that will induce $n+1$ distinct palettes. ■

Fig.1 shows the proper edge coloring α of the graph $Wd(2,6)$ inducing 3 distinct palettes.

We will also give an upper bound for the palette index of $Wd(n,k)$ for any k odd number.

Corollary 3. *For any positive integers $k, n \ge 2$, we have*

$$\check{s}(Wd(n,k)) \le \begin{cases} 2n+1, & \text{if } k \equiv 3 \pmod 4, \\ 3n+1, & \text{if } k \equiv 1 \pmod 4. \end{cases} \tag{3}$$

Proof. Suppose that K_k^i $(1 \le i \le n)$ are the copies of the complete graph in $Wd(n,k)$, and u is the shared universal vertex. Denote by $C_1, C_2, ..., C_n$ disjoint color-sets needed for a proper edge coloring of a complete graph that induces a minimum number of distinct palettes.

Case 1. $k \equiv 3 \pmod 4$. We will use the coloring described in the proof of [2](Proposition 5). Assume that $\forall i(1 \le i \le n)$ α_i is a proper edge coloring of K_k^i with color-set C_i inducing 3 distinct palettes. While constructing the α_i coloring, the complete graph's vertex set is partitioned into three sets. One of these sets is a one-element set, which induces a new unique palette. Taking $\{u\}$ as that set for any partition of $V(K_k^i)(1 \le i \le n)$ will let us obtain coloring of $Wd(n,k)$ that induces at most $2n+1$ distinct palettes.

Case 2. $k \equiv 1 \pmod 4$. We will use the coloring described in the proof of [2](Theorem 7). Assume that $\forall i(1 \le i \le n)$ α_i is a proper edge coloring of K_k^i with color-set C_i inducing 4 distinct palettes. In this case, coloring α_i also causes a new unique palette on the vertex of one-element set. Similar to the previous case, taking $\{u\}$ as that set for all partitions of $V(K_k^i)$ $(1 \le i \le n)$ will let us obtain coloring of $Wd(n,k)$ that induces $3n+1$ distinct palettes. ■

4. Conclusion

In the current article we examined the palette index of Cartesian products of graphs. Namely, we determined the palette index of the Cartesian product of cycles and paths and constructed colorings based on the length of the cycle, inducing a minimum number of palettes. Next, we gave some results connected to the palette index of the Cartesian product of complete graphs and paths. We also considered the problem of determining the palette index of windmill graphs. In particular, we showed the existence of coloring α, such that the number of palettes of $Wd(n, 2k)$ for any $n, k \geq 2$ induced by α is equal to $n + 1$. Moreover, we determined the upper bounds for the windmill graphs in case when the number of vertices of each complete graph is odd.

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Որոշ արդյունքներ գրաֆների դեկարտյան արտադրյալների պալիտրայի ինդեքսի մասին

Խաչիկ Ս. Սմբատյան

Երևանի պետական համալսարան
e-mail: smbatyan1729@gmail.com

Ամփոփում

Տրված է գրաֆի α-ճիշտ կողային ներկում, $S_G(v,\alpha)$-ով նշանակում են $v \in V(G)$ գագաթին կից կողերի բազմությունը: G ճիշտ կողային ներկման դեպքում իրարից տարբեր պալիտրաների նվազագույն քանակը անվանում են G- ի պալիտրային ինդեքս և նշանակում $\check{s}(G)$-ով: $Wd(n,k)$-ն չուղղորդված գրաֆ է, որը կառուցվում է $k \geq 2$ եվ $n \geq 2$ արժեքների համար n հատ K_k լրիվ գրաֆների մեկ ընդհանուր գագաթում միավորման միջոցով: Այս հոդվածում մենք կտանք պալիտրային ինդեքսի գնահատակն լրիվ գրաֆի եվ պարզ ճանապարհի դեկարդյան արտադրյալի համար: Մենք նաև տվել ենք պալիտրային ինդեքսի վերին գնահատականը $Wd(n,k)$-ի համար: Մասնավորապես, ցույց է տրվել, որ կամայական $k \geq 2$ եվ $n \geq 2$ զույգ ամբողջ թվերի համար $\check{s}(Wd(n,2k)) = n+1$:

Բանալի բառեր՝ Կողային ներկում, ճիշտ կողային ներկում, պալիտրա, պալիտրայի ինդեքս, Դեկարտյան արտադրյալ:

Некоторые результаты об индексе палитры декартово произведение графов

Хачик С. Смбатян

Ереванский государственный университет
e-mail: smbatyan1729@gmail.com

Аннотация

При правильной α-реберной раскраске графа G мы определяем палитру $S_G(v,\alpha)$ вершины $v \in V(G)$ как множество всех цветов, появляющихся на ребрах, смежных с v Индекс палитры $\check{s}(G)$ графа G является минимальным числом различных палитр, встречающихся при всех правильных реберных раскрасках G. В теории графов мельница $Wd(n,k)$- это неориентированный граф, построенный для $k \geq 2$ и $n \leq 2$ путём предприятий n копии полных графов K_k в одной общей вершине В этой статье мы даем оценку индекса палитры декартового произведения полных графов и простых путей. Мы также рассматриваем задачу определения индекса палитры графов мельниц. В частности, мы показываем, что для любых положительных целых чисел $k \geq 2$ и $n \leq 2$, $\check{s}(Wd(n,2k)) = n+1$.

Ключевые слова: реберная раскраска, правильная рёберная, раскраска, палитра, индекс палитры, декартово произведение, граф мельница.