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A Sharp Improvement of a Theorem of Bauer and Schmeichel

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Abstract

Let G be a graph on n vertices with minimum degree δ. The earliest nontrivial lower bound for the circumference c (the length of a longest cycle in G) was established in 1952 due to Dirac in terms of n and δ: (i) if G is a 2-connected graph, then $c \geq \min\{n, 2\delta\}$. The bound in Theorem (i) is sharp. In 1986, Bauer and Schmeichel gave a version of this classical result for 1-tough graphs: (ii) if G is a 1-tough graph, then $c \geq \min\{n, 2\delta+2\}$. In this paper we present an improvement of (ii), which is sharp for each n: (iii) if G is a 1-tough graph, then $c \geq \min\{n, 2\delta + 2\}$ when $n \equiv 1(mod\ 3)$; $c \geq \min\{n, 2\delta + 3\}$ when $n \equiv 2(mod\ 3)$ or $n \equiv 1(mod\ 4)$; and $c \geq \min\{n, 2\delta + 4\}$ otherwise.

Keywords: Hamilton cycle; circumference; minimum degree; 1-tough graphs.

1. Introduction

Throughout this article we consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. We use n, δ and c to denote the order of G, the minimum degree and the circumference - the length of a longest cycle in G, respectively. A good reference for any undefined terms is [2].

The earliest nontrivial lower bound for the circumference was established in 1952 due to Dirac [4] in terms of n and δ:

Theorem A [4]: *If G is a 2-connected graph, then* $c \geq \min\{n, 2\delta\}$.

The bound 2δ in Theorem A is sharp.

In 1973, Chvátal [3] introduced the concept of toughness. Since then a lot of research has been done towards finding the exact analogs of classical Hamiltonian results under additional 1-tough condition instead of 2-connectivity - an alternative and stronger necessary condition for a graph to be Hamiltonian.

The analog of the classical Theorem A for 1-tough graphs was established by Bauer and Schmeichel ([1], 1986).

Theorem B [1]: *If G is a 1-tough graph, then $c \geq \min\{n, 2\delta + 2\}$.*

The bound $2\delta+2$ in Theorem B is shown [1] to be sharp by constructing graphs of order $n \equiv 1(mod\ 3)$ with $c = 2\delta + 2$.

In this paper we show that the bound $2\delta + 2$ in Theorem B is sharp if and only if $n \equiv 1(mod\ 3)$. Furthermore, we present a sharp refinement of Theorem B, which is sharp for each n.

Theorem 1: *Every 1-tough graph is either Hamiltonian, or*

$$c \geq \begin{cases} 2\delta+2 & when \quad n \equiv 1(mod\ 3), \\ 2\delta+3 & when \quad n \equiv 2(mod\ 3) \quad or \quad n \equiv 1(mod\ 4), \\ 2\delta+4 & \qquad otherwise. \end{cases}$$

To see that Theorem 1 is sharp for each n, let $H_1, H_2, ..., H_h$ be disjoint complete graphs with distinct vertices $x_i, y_i \in V(H_i)$ $(i = 1, 2, ..., h)$. Form a new graph $H(t_1, t_2, ..., t_h)$ by identifying the vertices $x_1, x_2, ..., x_h$ and adding all possible edges between $y_1, y_2, ..., y_h$, where $t_i = |V(H_i)|$ $(i = 1, 2, ..., h)$. The graph $H(\delta+1, \delta+1, \delta+1)$ shows that the bound $2\delta + 2$ in Theorem 1 cannot be replaced by $2\delta + 3$ when $n \equiv 1(mod\ 3)$. Next, the graphs $H(\delta+2, \delta+1, \delta+1)$ and $H(\delta+1, \delta+1, \delta+1, \delta+1)$ show that the bound $2\delta+3$ cannot be replaced by $2\delta+4$ when $n \equiv 2(mod\ 3)$ or $n \equiv 1(mod\ 4)$. Finally, the graph $H(\delta+2, \delta+2, \delta+1)$ shows that the bound $2\delta + 4$ cannot be replaced by $2\delta + 5$.

2. Notations and Preliminaries

Let G be a graph. For S a subset of $V(G)$, we denote by $G\backslash S$ the maximum subgraph of G with vertex set $V(G)\backslash S$. We write $\langle S \rangle$ for the subgraph of G induced by S. For a subgraph H of G we use $G\backslash H$ short for $G\backslash V(H)$. The neighborhood and the degree of a vertex $x \in V(G)$ will be denoted by $N(x)$ and $d(x)$, respectively. Furthermore, for a subgraph H of G and $x \in V(G)$, we define $N_H(x) = N(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. Let $s(G)$ denote the number of components of a graph G. A graph G is 1-tough if $|S| \geq s(G\backslash S)$ for every subset S of the vertex set $V(G)$ with $s(G\backslash S) > 1$. A graph G on n vertices is Hamiltonian if G contains a Hamilton cycle, i.e., a cycle of length n.

Paths and cycles in a graph G are considered as subgraphs of G. If Q is a path or a cycle, then the length of Q, denoted by $|Q|$, is $|E(Q)|$. We write Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. We abbreviate x^{+1} and x^{-1} by x^+ and x^-, respectively. For each $X \subset V(C)$, we define $X^+ = \{x^+ | x \in X\}$ and $X^- = \{x^- | x \in X\}$.

Special definitions. Let G be a graph, C a longest cycle in G and $P = x\overrightarrow{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 0$. Let $\xi_1, \xi_2, ..., \xi_s$ be the elements of $N_C(x) \cup N_C(y)$ occurring on $\overrightarrow{C}$ in a consecutive order. Set

$$I_i = \xi_i\overrightarrow{C}\xi_{i+1}, \ I_i^* = \xi_i^+\overrightarrow{C}\xi_{i+1}^- \quad (i = 1, 2, ..., s),$$

where $\xi_{s+1} = \xi_1$.

(1) The segments $I_1, I_2, ..., I_s$ are called elementary segments on C induced by $N_C(x) \cup N_C(y)$.

(2) We call a path $L = z\overrightarrow{L}w$ an intermediate path between two distinct elementary segments I_a and I_b, if

$$z \in V(I_a^*),\ w \in V(I_b^*),\ V(L) \cap V(C \cup P) = \{z, w\}.$$

(3) Define $\Upsilon(I_{i_1}, I_{i_2}, ..., I_{i_t})$ to be the set of all intermediate paths between elementary segments $I_{i_1}, I_{i_2}, ..., I_{i_t}$.

(4) If $\Upsilon(I_1, ..., I_s) \subseteq E$, then the maximum number of intermediate independent edges (not having a common vertex) in $\Upsilon(I_1, ..., I_s)$ will be denoted by $\mu(\Upsilon)$.

(5) We say that two intermediate independent edges w_1w_2, w_3w_4 have a crossing, if either w_1, w_3, w_2, w_4 or w_1, w_4, w_2, w_3 occur on $\overrightarrow{C}$ in a consecutive order.

Lemma 1: *Let G be a graph, C a longest cycle in G and $P = x\overrightarrow{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 1$. If $|N_C(x)| \geq 2$, $|N_C(y)| \geq 2$ and $N_C(x) \neq N_C(y)$, then*

$$c \geq \begin{cases} 3\delta + \max\{\sigma_1, \sigma_2\} - 1 \geq 3\delta & \text{if} \quad \overline{p} = 1, \\ 4\delta - 2\overline{p} & \text{if} \quad \overline{p} \geq 2, \end{cases}$$

where $\sigma_1 = |N_C(x)\backslash N_C(y)|$ and $\sigma_2 = |N_C(y)\backslash N_C(x)|$.

Lemma 2: *Let G be a graph, C a longest cycle in G and $P = x\overrightarrow{P}y$ a longest path in $G\backslash C$ of length $\overline{p} \geq 0$. Let $N_C(x) = N_C(y)$, $|N_C(x)| \geq 2$ and $f, g \in \{1, ..., s\}$.*

(a1) If $L \in \Upsilon(I_f, I_g)$, then

$$|I_f| + |I_g| \geq 2\overline{p} + 2|L| + 4.$$

(a2) If $\Upsilon(I_f, I_g) \subseteq E(G)$ and $|\Upsilon(I_f, I_g)| = \varepsilon$ for some $\varepsilon \in \{1, 2, 3\}$, then

$$|I_f| + |I_g| \geq 2\overline{p} + \varepsilon + 5,$$

(a3) If $\Upsilon(I_f, I_g) \subseteq E(G)$ and $\Upsilon(I_f, I_g)$ contains two independent intermediate edges, then

$$|I_f| + |I_g| \geq 2\overline{p} + 8.$$

The following result is due to Voss [5].

Lemma 3 [5]: *Let G be a Hamiltonian graph, $\{v_1, v_2, ..., v_t\} \subseteq V(G)$ and $d(v_i) \geq t$ $(i = 1, 2, ..., t)$. Then each pair x, y of vertices of G is connected in G by a path of length at least t.*

3. Proofs

Proof of Lemma 1. Put

$$A_1 = N_C(x)\backslash N_C(y),\ A_2 = N_C(y)\backslash N_C(x),\ M = N_C(x)\cap N_C(y).$$

By the hypothesis, $N_C(x)\neq N_C(y)$, implying that

$$\max\{|A_1|,|A_2|\}\geq 1.$$

Let $\xi_1,\xi_2,...,\xi_s$ be the elements of $N_C(x)\cup N_C(y)$ occurring on $\overrightarrow{C}$ in a consecutive order. Put $I_i=\xi_i\overrightarrow{C}\xi_{i+1}$ $(i=1,2,...,s)$, where $\xi_{s+1}=\xi_1$. Clearly, $s=|A_1|+|A_2|+|M|$. Since C is extreme, we have $|I_i|\geq 2$ $(i=1,2,...,s)$. Next, if $\{\xi_i,\xi_{i+1}\}\cap M\neq\emptyset$ for some $i\in\{1,2,...,s\}$, then $|I_i|\geq\overline{p}+2$. Further, if either $\xi_i\in A_1$, $\xi_{i+1}\in A_2$ or $\xi_i\in A_2$, $\xi_{i+1}\in A_1$, then again $|I_i|\geq\overline{p}+2$.

Case 1. $\overline{p}=1$.

Case 1.1. $|A_i|\geq 1$ $(i=1,2)$.

It follows that among $I_1,I_2,...,I_s$ there are $|M|+2$ segments of length at least $\overline{p}+2$. Observing also that each of the remaining $s-(|M|+2)$ segments has a length at least 2, we have

$$c\geq(\overline{p}+2)(|M|+2)+2(s-|M|-2)$$
$$=3(|M|+2)+2(|A_1|+|A_2|-2)=2|A_1|+2|A_2|+3|M|+2.$$

Since $|A_1|=d(x)-|M|-1$ and $|A_2|=d(y)-|M|-1$, we have

$$c\geq 2d(x)+2d(y)-|M|-2\geq 3\delta+d(x)-|M|-2.$$

Recalling that $d(x)=|M|+|A_1|+1$, we get

$$c\geq 3\delta+|A_1|-1=3\delta+\sigma_1-1.$$

Analogously, $c\geq 3\delta+\sigma_2-1$. So,

$$c\geq 3\delta+\max\{\sigma_1,\sigma_2\}-1\geq 3\delta.$$

Case 1.2. Either $|A_1|\geq 1,|A_2|=0$ or $|A_1|=0,|A_2|\geq 1$.

Assume w.l.o.g. that $|A_1|\geq 1$ and $|A_2|=0$, i.e. $|N_C(y)|=|M|\geq 2$ and $s=|A_1|+|M|$. Hence, among $I_1,I_2,...,I_s$ there are $|M|+1$ segments of length at least $\overline{p}+2=3$. Taking into account that $|M|+1=d(y)$ and each of the remaining $s-(|M|+1)$ segments has a length at least 2, we get

$$c\geq 3(|M|+1)+2(s-|M|-1)=3d(y)+2(|A_1|-1)$$
$$\geq 3\delta+|A_1|-1=3\delta+\max\{\sigma_1,\sigma_2\}-1\geq 3\delta.$$

Case 2. $\overline{p}\geq 2$.

Case 2.1. $|A_i|\geq 1$ $(i=1,2)$.

It follows that among $I_1,I_2,...,I_s$ there are $|M|+2$ segments of length at least $\overline{p}+2$. Further, since each of the remaining $s-(|M|+2)$ segments has a length at least 2, we get

$$c\geq(\overline{p}+2)(|M|+2)+2(s-|M|-2)$$

$$= (\overline{p}-2)|M| + (2\overline{p}+4|M|+4) + 2(|A_1|+|A_2|-2)$$
$$\geq 2|A_1| + 2|A_2| + 4|M| + 2\overline{p}.$$

Observing also that

$$|A_1| + |M| + \overline{p} \geq d(x), \quad |A_2| + |M| + \overline{p} \geq d(y),$$

we have

$$2|A_1| + 2|A_2| + 4|M| + 2\overline{p}$$
$$\geq 2d(x) + 2d(y) - 2\overline{p} \geq 4\delta - 2\overline{p},$$

implying that $c \geq 4\delta - 2\overline{p}$.

Case 2.2. Either $|A_1| \geq 1, |A_2| = 0$ or $|A_1| = 0, |A_2| \geq 1$.

Assume w.l.o.g. that $|A_1| \geq 1$ and $|A_2| = 0$, that is $|N_C(y)| = |M| \geq 2$ and $s = |A_1|+|M|$. It follows that among $I_1, I_2, ..., I_s$ there are $|M|+1$ segments of length at least $\overline{p}+2$. Observing also that $|M| + \overline{p} \geq d(y) \geq \delta$, i.e., $2\overline{p} + 4|M| \geq 4\delta - 2\overline{p}$, we get

$$c \geq (\overline{p}+2)(|M|+1) \geq (\overline{p}-2)(|M|-1) + 2\overline{p} + 4|M|$$
$$\geq 2\overline{p} + 4|M| \geq 4\delta - 2\overline{p}. \qquad \blacksquare$$

Proof of Lemma 2. Let $\xi_1, \xi_2, ..., \xi_s$ be the elements of $N_C(x)$ occurring on $\overrightarrow{C}$ in a consecutive order. Put $I_i = \xi_i \overrightarrow{C} \xi_{i+1}$ $(i = 1, 2, ..., s)$, where $\xi_{s+1} = \xi_1$. To prove $(a1)$, let $L \in \Upsilon(I_f, I_g)$. Further, let $L = z\overrightarrow{L}w$ with $z \in V(I_f^*)$ and $w \in V(I_g^*)$. Put

$$|\xi_f \overrightarrow{C} z| = d_1, \ |z\overrightarrow{C}\xi_{f+1}| = d_2, \ |\xi_g \overrightarrow{C} w| = d_3, \ |w\overrightarrow{C}\xi_{g+1}| = d_4,$$

$$C' = \xi_f x \overrightarrow{P} y \xi_g \overleftarrow{C} z \overrightarrow{L} w \overrightarrow{C} \xi_f.$$

Clearly,

$$|C'| = |C| - d_1 - d_3 + |L| + |P| + 2.$$

Since C is extreme, we have $|C| \geq |C'|$, implying that $d_1 + d_3 \geq \overline{p} + |L| + 2$. By a symmetric argument, $d_2 + d_4 \geq \overline{p} + |L| + 2$. Hence

$$|I_f| + |I_g| = \sum_{i=1}^{4} d_i \geq 2\overline{p} + 2|L| + 4.$$

The proof of $(a1)$ is complete. To prove $(a2)$ and $(a3)$, let $\Upsilon(I_f, I_g) \subseteq E(G)$ and $|\Upsilon(I_f, I_g)| = \varepsilon$ for some $\varepsilon \in \{1, 2, 3\}$.

Case 1. $\varepsilon = 1$.

Let $L \in \Upsilon(I_f, I_g)$, where $|L| = 1$. By (a1),

$$|I_f| + |I_g| \geq 2\overline{p} + 2|L| + 4 = 2\overline{p} + 6.$$

Case 2. $\varepsilon = 2$.

It follows that $\Upsilon(I_f, I_g)$ consists of two edges e_1, e_2. Put $e_1 = z_1w_1$ and $e_2 = z_2w_2$, where $\{z_1, z_2\} \subseteq V(I_f^*)$ and $\{w_1, w_2\} \subseteq V(I_g^*)$.

Case 2.1. $z_1 \neq z_2$ and $w_1 \neq w_2$.
Assume w.l.o.g. that z_1 and z_2 occur in this order on I_f.

Case 2.1.1. w_2 and w_1 occur in this order on I_g.
Put

$$|\xi_f \overrightarrow{C} z_1| = d_1,\ |z_1 \overrightarrow{C} z_2| = d_2,\ |z_2 \overrightarrow{C} \xi_{f+1}| = d_3,$$

$$|\xi_g \overrightarrow{C} w_2| = d_4,\ |w_2 \overrightarrow{C} w_1| = d_5,\ |w_1 \overrightarrow{C} \xi_{g+1}| = d_6,$$

$$C' = \xi_f \overrightarrow{C} z_1 w_1 \overleftarrow{C} w_2 z_2 \overrightarrow{C} \xi_g x \overrightarrow{P} y \xi_{g+1} \overrightarrow{C} \xi_f.$$

Clearly,

$$|C'| = |C| - d_2 - d_4 - d_6 + |\{e_1\}| + |\{e_2\}| + |P| + 2$$
$$= |C| - d_2 - d_4 - d_6 + \overline{p} + 4.$$

Since C is extreme, we have $|C| \geq |C'|$, implying that $d_2 + d_4 + d_6 \geq \overline{p} + 4$. By a symmetric argument, $d_1 + d_3 + d_5 \geq \overline{p} + 4$. Hence

$$|I_f| + |I_g| = \sum_{i=1}^{6} d_i \geq 2\overline{p} + 8.$$

Case 2.1.2. w_1 and w_2 occur in this order on I_g.
Putting

$$C' = \xi_f \overrightarrow{C} z_1 w_1 \overrightarrow{C} w_2 z_2 \overrightarrow{C} \xi_g x \overrightarrow{P} y \xi_{g+1} \overrightarrow{C} \xi_f,$$

we can argue as in Case 2.1.1.

Case 2.2. Either $z_1 = z_2$, $w_1 \neq w_2$ or $z_1 \neq z_2$, $w_1 = w_2$.
Assume w.l.o.g. that $z_1 \neq z_2$, $w_1 = w_2$ and z_1, z_2 occur in this order on I_f. Put

$$|\xi_f \overrightarrow{C} z_1| = d_1,\ |z_1 \overrightarrow{C} z_2| = d_2,\ |z_2 \overrightarrow{C} \xi_{f+1}| = d_3,$$

$$|\xi_g \overrightarrow{C} w_1| = d_4,\ |w_1 \overrightarrow{C} \xi_{g+1}| = d_5,$$

$$C' = \xi_f x \overrightarrow{P} y \xi_g \overleftarrow{C} z_1 w_1 \overrightarrow{C} \xi_f,$$

$$C'' = \xi_f \overrightarrow{C} z_2 w_1 \overleftarrow{C} \xi_{f+1} x \overrightarrow{P} y \xi_{g+1} \overrightarrow{C} \xi_f.$$

Clearly,

$$|C'| = |C| - d_1 - d_4 + |\{e_1\}| + |P| + 2 = |C| - d_1 - d_4 + \overline{p} + 3,$$

$$|C''| = |C| - d_3 - d_5 + |\{e_2\}| + |P| + 2 = |C| - d_3 - d_5 + \overline{p} + 3.$$

Since C is extreme, $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_4 \geq \overline{p} + 3,\ d_3 + d_5 \geq \overline{p} + 3.$$

Hence,

$$|I_f| + |I_g| = \sum_{i=1}^{5} d_i \geq d_1 + d_3 + d_4 + d_5 + 1 \geq 2\overline{p} + 7.$$

Case 3. $\varepsilon = 3$.

It follows that $\Upsilon(I_f, I_g)$ consists of three edges e_1, e_2, e_3. Let $e_i = z_i w_i$ $(i = 1, 2, 3)$, where $\{z_1, z_2, z_3\} \subseteq V(I_f^*)$ and $\{w_1, w_2, w_3\} \subseteq V(I_g^*)$. If there are two independent edges among e_1, e_2, e_3, then we can argue as in Case 2.1. Otherwise, we can assume w.l.o.g. that $w_1 = w_2 = w_3$ and z_1, z_2, z_3 occur in this order on I_f. Put

$$|\xi_f \overrightarrow{C} z_1| = d_1,\ |z_1 \overrightarrow{C} z_2| = d_2,\ |z_2 \overrightarrow{C} z_3| = d_3,$$

$$|z_3 \overrightarrow{C} \xi_{f+1}| = d_4,\ |\xi_g \overrightarrow{C} w_1| = d_5,\ |w_1 \overrightarrow{C} \xi_{g+1}| = d_6,$$

$$C' = \xi_f x \overrightarrow{P} y \xi_g \overleftarrow{C} z_1 w_1 \overrightarrow{C} \xi_f,$$

$$C'' = \xi_f \overrightarrow{C} z_3 w_1 \overleftarrow{C} \xi_{f+1} x \overrightarrow{P} y \xi_{g+1} \overrightarrow{C} \xi_f.$$

Clearly,

$$|C'| = |C| - d_1 - d_5 + |\{e_1\}| + \overline{p} + 2,$$

$$|C''| = |C| - d_4 - d_6 + |\{e_3\}| + \overline{p} + 2.$$

Since C is extreme, we have $|C| \geq |C'|$ and $|C| \geq |C''|$, implying that

$$d_1 + d_5 \geq \overline{p} + 3,\ d_4 + d_6 \geq \overline{p} + 3.$$

Hence,

$$|I_f| + |I_g| = \sum_{i=1}^{6} d_i \geq d_1 + d_4 + d_5 + d_6 + 2 \geq 2\overline{p} + 8. \qquad \blacksquare$$

Proof of Theorem 1. Let G be a 1-tough graph. If $c \geq 2\delta + 4$, then we are done. Hence, we can assume that

$$c \leq 2\delta + 3. \tag{1}$$

Let C be a longest cycle in G and $P = x_1 \overrightarrow{P} x_2$ a longest path in $G \backslash C$. Put $|P| = |V(P)| - 1 = \overline{p}$. If $|V(P)| = 0$, then C is a Hamilton cycle and we are done. Let $|V(P)| \geq 1$, that is $\overline{p} \geq 0$. Put $X = N_C(x_1) \cup N_C(x_2)$ and let $\xi_1, ..., \xi_s$ be the elements of X occurring on C in a consecutive order. Put

$$I_i = \xi_i \overrightarrow{C} \xi_{i+1},\ I_i^* = \xi_i^+ \overrightarrow{C} \xi_{i+1}^- \quad (i = 1, ..., s),$$

where $\xi_{s+1} = \xi_1$. Since G is a 1-tough graph, we have $\delta \geq 2$.

Case 1. $\overline{p} \leq \delta - 2$.

It follows that $s \geq |N_C(x_i)| \geq \delta - \overline{p} \geq 2$ $(i = 1, 2)$. Assume first that $N_C(x_1) \neq N_C(x_2)$, implying that $\overline{p} \geq 1$. If $\overline{p} \geq 2$, then by Lemma 1, $c \geq 4\delta - 2\overline{p} \geq 2\delta + 4$, contradicting (1). Hence $\overline{p} = 1$, which yields $\delta \geq \overline{p} + 2 = 3$. By Lemma 1, $c \geq 3\delta \geq 9$. If $\delta \geq 4$, then $c \geq 3\delta \geq 2\delta + 4$, contradicting (1). Let $\delta = 3$. Next, we can suppose that $c = 9$, since otherwise $c \geq 10 = 3\delta + 1 = 2\delta + 4$, contradicting (1). Further, we can suppose that $s \geq 3$, since $N_C(x_1) = N_C(x_2)$ when $s = 2$, contradicting the hypothesis. Finally, we can suppose that $s=3$, since clearly $c\geq 10$ when $s\geq 4$, a contradiction. Thus, $|I_1|=|I_2|=|I_3|=3$ and it is not hard to see that $G\backslash\{\xi_1,\xi_2,\xi_3\}$ has at least four components, contradicting $\tau\geq 1$.

Now assume that $N_C(x_1)=N_C(x_2)$. Since C is extreme, we have

$$|I_i|\geq|\xi_i x_1\overrightarrow{P}x_2\xi_{i+1}|\geq\overline{p}+2\ \ (i=1,...,s).$$

Case 1.1. $s\geq\delta-\overline{p}+1$.

Clearly,

$$c=\sum_{i=1}^{s}|I_i|\geq s(\overline{p}+2)$$

$$\geq(\delta-\overline{p}+1)(\overline{p}+2)=(\delta-\overline{p}-2)\overline{p}+2\delta+\overline{p}+2. \tag{2}$$

If $\overline{p}\geq 2$, then by (2), $c\geq 2\delta+4$, contradicting (1). Let $\overline{p}\leq 1$.

Case 1.1.1. $\overline{p}=0$.

If $\Upsilon(I_1,...,I_s)=\emptyset$, then $G\backslash\{\xi_1,...,\xi_s\}$ has at least $s+1$ components, contradicting the fact that $\tau\geq 1$. Otherwise $\Upsilon(I_a,I_b)\neq\emptyset$ for some distinct $a,b\in\{1,...,s\}$. Let $L\in\Upsilon(I_a,I_b)$. By Lemma 2(a1),

$$|I_a|+|I_b|\geq 2\overline{p}+2|L|+4\geq 6.$$

Recalling also that $s\geq\delta-\overline{p}+1=\delta+1$, we get

$$c=\sum_{i=1}^{s}|I_i|\geq|I_a|+|I_b|+2(s-2)=2s+2\geq 2\delta+4,$$

contradicting (1).

Case 1.1.2. $\overline{p}=1$.

By (2), $c\geq 3\delta$. We can suppose that $\delta\leq 3$, since $c\geq 3\delta\geq 2\delta+4$ when $\delta\geq 4$, contradicting (1). On the other hand, by the hypothesis, $\delta\geq\overline{p}+2=3$, implying that $\delta=3$. By the hypothesis, $s\geq\delta-\overline{p}+1=3$. Next, we can suppose that $s=3$, since $c\geq s(\overline{p}+2)\geq 12=2\delta+6$ when $s\geq 4$, contradicting (1). Further, if $\Upsilon(I_1,I_2,I_3)=\emptyset$, then $G\backslash\{\xi_1,\xi_2,\xi_3\}$ has at least four components, contradicting $\tau\geq 1$. Otherwise $\Upsilon(I_a,I_b)\neq\emptyset$ for some distinct $a,b\in\{1,2,3\}$, say $a=1$ and $b=2$. Let $L\in\Upsilon(I_1,I_2)$. By Lemma 2(a1),

$$|I_1|+|I_2|\geq 2\overline{p}+2|L|+4=8,$$

which yields $c\geq|I_1|+|I_2|+|I_3|\geq 11=2\delta+5$, contradicting (1).

Case 1.2. $s=\delta-\overline{p}$.

It follows that $x_1x_2\in E$. Then $x_1x_2\overleftarrow{P}x_1^+$ is another longest path in $G\backslash C$. We can suppose that $N_C(x_1)=N_C(x_1^+)$, since otherwise we can argue as in Case 1. By the same reason,

$$N_C(x_1)=N_C(x_1^+)=N_C(x_1^{+2})=...=N_C(x_2).$$

Since C is extreme, we have $|I_i|\geq|\xi_i x_1\overrightarrow{P}x_2\xi_{i+1}|=\overline{p}+2$ $(i=1,...,s)$. If $\Upsilon(I_1,...,I_s)=\emptyset$, then $G\backslash\{\xi_1,...,\xi_s\}$ has at least $s+1$ components, contradicting $\tau\geq 1$. Otherwise $\Upsilon(I_a,I_b)\neq$

$\emptyset$ for some distinct $a, b \in \{1, ..., s\}$. Let $L \in \Upsilon(I_a, I_b)$ with $L = z_1 \overrightarrow{L} z_2$, where $z_1 \in V(I_a^*)$ and $z_2 \in V(I_b^*)$. By Lemma 2(a1), $|I_a| + |I_b| \geq 2\overline{p} + 6$. Hence

$$c = \sum_{i=1}^{s} |I_i| \geq |I_a| + |I_b| + (s-2)(\overline{p}+2) \geq s(\overline{p}+2) + 2$$

$$= (\delta - \overline{p})(\overline{p}+2) + 2 = 2\delta + 2 + \overline{p}(\delta - \overline{p} - 2). \quad (3)$$

Claim 1. $(a1)$ $2\overline{p} + 6 \leq |I_a| + |I_b| \leq 2\overline{p} + 7$ and $|I_i| \leq \overline{p} + 5$ $(i = 1, ..., s)$.

$(a2)$ If $|I_a| + |I_b| = 2\overline{p} + 7$, then $|I_i| = \overline{p} + 2$ for each $i \in \{1, ..., s\} \backslash \{a, b\}$.

$(a3)$ If $|I_a| + |I_b| = 2\overline{p} + 6$, then $|I_f| \leq \overline{p} + 3$ for some $f \in \{1, ..., s\} \backslash \{a, b\}$ and $|I_i| = \overline{p} + 2$ for each $i \in \{1, ..., s\} \backslash \{a, b, f\}$.

$(a4)$ If $|I_f| = \overline{p} + 5$ for some $f \in \{a, b\}$, then $|I_i| = \overline{p} + 2$ for each $i \in \{1, ..., s\} \backslash \{f\}$.

$(a5)$ For each distinct $f, g, h \in \{1, ..., s\}$, $|I_f| + |I_g| + |I_h| \leq 3\overline{p} + 9$.

$(a6)$ $\Upsilon(I_1, ..., I_s) \subseteq E$.

Proof. If $|I_f| \geq \overline{p} + 6$ for some $f \in \{1, ..., s\}$, then

$$c = \sum_{i=1}^{s} |I_i| \geq |I_f| + (s-1)(\overline{p}+2) \geq s(\overline{p}+2) + 4$$

$$= 2\delta + 4 + \overline{p}(\delta - \overline{p} - 2) \geq 2\delta + 4,$$

contradicting (1). Next, if $|I_a| + |I_b| \geq 2\overline{p} + 8$, then

$$c \geq |I_a| + |I_b| + (s-2)(\overline{p}+2) \geq s(\overline{p}+2) + 4 \geq 2\delta + 4,$$

again contradicting (1). Hence $(a1)$ holds. Statements $(a2) - (a4)$ can be proved by a similar way. To prove $(a5)$, assume the contrary, that is $|I_f| + |I_g| + |I_h| \geq 3\overline{p} + 10$ for some distinct $f, g, h \in \{1, ..., s\}$. Then

$$c = \sum_{i=1}^{s} |I_i| \geq |I_f| + |I_g| + |I_h| + (s-3)(\overline{p}+2)$$

$$\geq 3(\overline{p}+2) + 4 + (s-3)(\overline{p}+2) = 2\delta + 4 + \overline{p}(s-2) \geq 2\delta + 4,$$

contradicting (1). Statement $(a6)$ follows from Lemma 2(a1) and Claim 1(a1). Claim 1 is proved.

Claim 2. $\overline{p} + 3 \leq d_1 \leq \overline{p} + 4$ and $\overline{p} + 3 \leq d_2 \leq \overline{p} + 4$, where

$$d_1 = |\xi_a \overrightarrow{C} z_1| + |\xi_b \overrightarrow{C} z_2|, \quad d_2 = |z_1 \overrightarrow{C} \xi_{a+1}| + |z_2 \overrightarrow{C} \xi_{b+1}|.$$

Proof. Put

$$Q = \xi_a x_1 \overrightarrow{P} x_2 \xi_b \overleftarrow{C} z_1 z_2 \overrightarrow{C} \xi_a.$$

Clearly, $|Q| = |C| - d_1 + \overline{p} + 3$. Since C is extreme, we have $|C| \geq |Q|$, implying that $d_1 \geq \overline{p} + 3$. By a symmetric argument, $d_2 \geq \overline{p} + 3$. By Claim 1(a1), $|I_a| + |I_b| = d_1 + d_2 \leq 2\overline{p} + 7$. If $d_1 \geq \overline{p} + 5$, then $2\overline{p} + 7 \geq d_1 + d_2 \geq \overline{p} + 5 + d_2$, implying that $d_2 \leq \overline{p} + 2$, a contradiction. Hence, $d_1 \leq \overline{p} + 4$. By a symmetric argument, $d_2 \leq \overline{p} + 4$. Claim 2 is proved.

Claim 3. If $v_1 \in V(\xi_a^+ \overrightarrow{C} z_1^-)$ and $v_2 \in V(z_1^+ \overrightarrow{C} \xi_{a+1}^-)$, then $v_1 v_2 \notin E$.

Proof. Assume the contrary, that is $v_1v_2 \in E$. Put

$$Q = \xi_a \overrightarrow{C} v_1 v_2 \overleftarrow{C} z_1 z_2 \overleftarrow{C} \xi_{a+1} x_1 \overrightarrow{P} x_2 \xi_{b+1} \overrightarrow{C} \xi_a,$$

$$|\xi_a \overrightarrow{C} v_1| = d_1, \quad |v_1 \overrightarrow{C} z_1| = d_2, \quad |z_1 \overrightarrow{C} v_2| = d_3,$$

$$|v_2 \overrightarrow{C} \xi_{a+1}| = d_4, \quad |\xi_b \overrightarrow{C} z_2| = d_5, \quad |z_2 \overrightarrow{C} \xi_{b+1}| = d_6.$$

Clearly, $|Q| = |C| - d_2 - d_4 - d_6 + \overline{p} + 4$. Since C is extreme, we have $|Q| \le |C|$, implying that $d_2 + d_4 + d_6 \ge \overline{p} + 4$. By a symmetric argument, $d_1 + d_3 + d_5 \ge \overline{p} + 4$. By summing, we get

$$\sum_{i=1}^{6} d_i = |I_a| + |I_b| \ge 2\overline{p} + 8,$$

contradicting Claim 1(a1). Thus, $v_1v_2 \notin E$. Claim 3 is proved.

Claim 4. Let ξ_f, ξ_g, ξ_h occur on $\overrightarrow{C}$ in a consecutive order for some $f, g, h \in \{1, ..., s\}$ and $w_1w_2 \in E$ for some $w_1 \in V(I_f^*)$ and $w_2 \in V(I_g^*)$. If $N(w_3) \cap \{\xi_{f+1}, \xi_g\} \neq \emptyset$ for some $w_3 \in V(I_h^*)$, then

$$|w_1 \overrightarrow{C} \xi_{f+1}| + |\xi_g \overrightarrow{C} w_2| + |\xi_h \overrightarrow{C} w_3| \ge \overline{p} + 4.$$

Further, if $N(w_4) \cap \{\xi_{f+1}, \xi_g\} \neq \emptyset$ for some $w_4 \in V(I_{h-1}^*)$, then

$$|w_1 \overrightarrow{C} \xi_{f+1}| + |\xi_g \overrightarrow{C} w_2| + |w_4 \overrightarrow{C} \xi_h| \ge \overline{p} + 4.$$

Proof. Assume first that $w_3\xi_{f+1} \in E$. Put

$$Q = \xi_f \overrightarrow{C} w_1 w_2 \overrightarrow{C} \xi_h x_1 \overrightarrow{P} x_2 \xi_g \overleftarrow{C} \xi_{f+1} w_3 \xi_f.$$

Clearly,

$$|Q| = |C| - |w_1 \overrightarrow{C} \xi_{f+1}| - |\xi_g \overrightarrow{C} w_2| - |\xi_h \overrightarrow{C} w_3| + \overline{p} + 4.$$

Since $|Q| \le |C|$, the desired result holds immediately. If $w_4\xi_{f+1} \in E$, then we can use the following cycle

$$Q' = \xi_f \overrightarrow{C} w_1 w_2 \overrightarrow{C} w_4 \xi_{f+1} \overrightarrow{C} \xi_g x_2 \overleftarrow{P} x_1 \xi_h \overrightarrow{C} \xi_f$$

instead of Q. By a symmetric argument, the desired result holds when either $w_3\xi_g \in E$ or $w_4\xi_g \in E$. Claim 4 is proved.

Claim 5. Every two intermediate independent edges e_1, e_2 in $\Upsilon(I_1, ..., I_s)$ have a crossing with $e_1, e_2 \in \Upsilon(I_f, I_g, I_h)$ for some distinct $f, g, h \in \{1, ..., s\}$.

Proof. Let $e_1 = w_1w_2$ and $e_2 = w_3w_4$. We distinguish three different cases. First, if $e_1, e_2 \in \Upsilon(I_f, I_g)$ for some distinct f, g, then by Lemma 2(a3), $|I_f| + |I_g| \ge 2\overline{p} + 8$, contradicting Claim 1(a1). Next, if $e_1 \in \Upsilon(I_f, I_g)$ and $e_2 \in \Upsilon(I_h, I_r)$ for some distinct f, g, h, r, then by Lemma 2(a1), $|I_f| + |I_g| \ge 2\overline{p} + 6$ and $|I_h| + |I_r| \ge 2\overline{p} + 6$, implying that

$$c \ge |I_f| + |I_g| + |I_h| + |I_r| + (s-4)(\overline{p}+2) = 4\overline{p} + 12 + (s-4)(\overline{p}+2)$$

$$= s(\overline{p}+2) + 4 = 2\delta + 4 + \overline{p}(\delta - \overline{p} - 2) \ge 2\delta + 4,$$

which again contradicts (1). Finally, let $e_1 \in \Upsilon(I_f, I_g)$ and $e_2 \in \Upsilon(I_f, I_h)$ for some distinct f, g, h. Assume w.l.o.g. that ξ_f, ξ_g, ξ_h occur on $\overrightarrow{C}$ in a consecutive order and $w_1, w_3 \in V(I_f^*)$,

$w_2 \in V(I_g^*)$, $w_4 \in V(I_h^*)$. We can assume also that w_3 and w_1 occur on I_f in a consecutive order, since otherwise e_1 and e_2 have a crossing and we are done. Put

$$Q = \xi_f \overrightarrow{C} w_3 w_4 \overleftarrow{C} w_2 w_1 \overrightarrow{C} \xi_g x_2 \overleftarrow{P} x_1 \xi_{h+1} \overrightarrow{C} \xi_f,$$

$$|\xi_f \overrightarrow{C} w_3| = d_1, \ |w_3 \overrightarrow{C} w_1| = d_2, \ |w_1 \overrightarrow{C} \xi_{f+1}| = d_3,$$

$$|\xi_g \overrightarrow{C} w_2| = d_4, \ |w_2 \overrightarrow{C} \xi_{g+1}| = d_5, \ |\xi_h \overrightarrow{C} w_4| = d_6, \ |w_4 \overrightarrow{C} \xi_{h+1}| = d_7.$$

Clearly, $|Q| = |C| - d_2 - d_4 - d_7 + \overline{p} + 4$. Since C is extreme, we have $|Q| \le |C|$, implying that $d_2 + d_4 + d_7 \ge \overline{p} + 4$. On the other hand, by Lemma 2, $d_3 + d_5 \ge \overline{p} + 3$ and $d_1 + d_6 \ge \overline{p} + 3$. By summing, we get $\sum_{i=1}^{7} d_i = |I_f| + |I_g| + |I_h| \ge 3\overline{p} + 10$. Then

$$|C| \ge |I_f| + |I_g| + |I_h| + (s-3)(\overline{p}+2) = s(\overline{p}+2) + 4 \ge 2\delta + 4,$$

contradicting (1). Claim 5 is proved.

Claim 6. If $\mu(\Upsilon) = 1$, then $s \le 3$ and either $\xi_a^+ \xi_{b+1}^- \in E$ with $\xi_a = \xi_{b+1}$ or $\xi_{a+1}^- \xi_b^+ \in E$ with $\xi_{a+1} = \xi_b$. If $\mu(\Upsilon) = 1$ and $s = 3$, then $|I_1| = |I_2| = |I_3| = \overline{p} + 3$.

Proof. Since $\mu(\Upsilon) = 1$, either one of the vertices z_1, z_2, say z_1, is a common vertex for all edges in $\Upsilon(I_1, ..., I_s)$ or $z_1 z_3, z_2 z_3 \in \Upsilon(I_1, ..., I_s)$ for some $z_3 \in V(I_f^*)$ and $f \in \{1, ..., s\} \backslash \{a, b\}$.

Case a1. z_1 is a common vertex for all edges in $\Upsilon(I_1, ..., I_s)$.

If $z_1 \notin \{\xi_a^+, \xi_{a+1}^-\}$, then by Claim 3, $G \backslash \{\xi_1, ..., \xi_s, z_1\}$ has at least $s+2$ components, contradicting $\tau \ge 1$. Let $z_1 \in \{\xi_a^+, \xi_{a+1}^-\}$, say $z_1 = \xi_a^+$.

Case a1.1. $z_1 \xi_{b+1}^- \notin E$.

It follows that $z_2 \ne \xi_{b+1}^-$. By Claim 2, $|\xi_b \overrightarrow{C} z_2| \ge \overline{p} + 2$.

Case a1.1.1. $z_1 \xi_{b+1}^{-2} \notin E$.

It follows that $|I_b| \ge \overline{p} + 5$. By Claim 1(a1), $|I_a| = \overline{p} + 2$. Moreover, we have $|I_b| = \overline{p} + 5$, $|\xi_b \overrightarrow{C} z_2| = \overline{p} + 2$, $z_2 = \xi_{b+1}^{-3}$ and $N(z_1) \cap V(I_b^*) = \{z_2\}$. By Claim 1(a4), $|I_i| = \overline{p} + 2$ for each $i \in \{1, ..., s\} \backslash \{b\}$. Next, by Lemma 2(a1), $\Upsilon(I_a, I_i) = \emptyset$ for each $i \in \{1, ..., s\} \backslash \{a, b\}$. Thus, if $z_1 y \in \Upsilon(I_1, ..., I_s)$, then $y = z_2$, implying that $\Upsilon(I_1, ..., I_s) = \{z_1 z_2\}$. Besides, since $|\xi_b \overrightarrow{C} z_2| = \overline{p} + 2 \ge 2$, we have $z_2 \notin \{\xi_b^+, \xi_{b+1}^-\}$. Therefore, by Claim 3, $G \backslash \{\xi_1, ..., \xi_s, z_2\}$ has at least $s+2$ components, contradicting $\tau \ge 1$.

Case a1.1.2. $z_1 \xi_{b+1}^{-2} \in E$.

It follows that $|I_b| \ge \overline{p} + 4$. Assume first that $|I_b| = \overline{p} + 5$. If $z_1 \xi_{b+1}^{-3} \notin E$, then clearly $z_2 = \xi_{b+1}^{-2}$ and we can argue as in Case a1.1.1. Otherwise the following cycle

$$\xi_a x_1 \overrightarrow{P} x_2 \xi_{a+1} \overrightarrow{C} \xi_{b+1}^{-3} z_1 \xi_{b+1}^{-2} \overrightarrow{C} \xi_a$$

is longer than C, a contradiction.

Now assume that $|I_b| = \overline{p} + 4$, that is $|\xi_b \overrightarrow{C} \xi_{b+1}^{-2}| = \overline{p} + 2$. If $z_1 y \in E$ for some $y \in V(\xi_b \overrightarrow{C} \xi_{b+1}^{-3})$, then by Claim 2, $|\xi_b \overrightarrow{C} y| \ge \overline{p} + 2$, implying that $|I_b| \ge \overline{p} + 5$, a contradiction. Hence, if $z_1 y \in \Upsilon(I_a, I_b)$, then clearly $y = \xi_{b+1}^{-2}$. In particular, we have $z_2 = \xi_{b+1}^{-2}$. Further, if $z_1 y \in \Upsilon(I_a, I_f)$ for some $f \in \{1, ..., s\} \backslash \{b\}$, then by Lemma 2(a1), $|I_a| + |I_f| \ge 2\overline{p} + 6$, that is $|I_a|+|I_b|+|I_f|\geq 3\overline{p}+10$, contradicting Claim 1(a5). Thus, z_2 is a common vertex for all edges in $\Upsilon(I_1,...,I_s)$. By Claim 3, $G\backslash\{\xi_1,...,\xi_s,z_2\}$ has at least $s+2$ components, contradicting $\tau\geq 1$.

Case a1.2. $\xi_a^+\xi_{b+1}^-\in E$.

By Claim 2, $|\xi_a^+\overrightarrow{C}\xi_{a+1}|\geq\overline{p}+2$ and $|\xi_b\overrightarrow{C}\xi_{b+1}^-|\geq\overline{p}+2$. If $|\xi_a^+\overrightarrow{C}\xi_{a+1}|\geq\overline{p}+3$ and $|\xi_b\overrightarrow{C}\xi_{b+1}^-|\geq\overline{p}+3$, then $|I_a|+|I_b|\geq 2\overline{p}+8$, contradicting Claim 1(a1). Hence, we can assume w.l.o.g. that $|\xi_b\overrightarrow{C}\xi_{b+1}^-|=\overline{p}+2$, that is $|I_b|=\overline{p}+3$ and $|I_a|\geq\overline{p}+3$. Further, we have $\xi_b^+\xi_a,\xi_b^+\xi_{b+1}\notin E$ (by Claim 4) and $\xi_b^+\xi_a^+\notin E$ (by Claim 2).

Case a1.2.1. $N(\xi_b^+)\nsubseteq V(C)$.

Let $Q=\xi_b^+\overrightarrow{Q}v$ be a longest path in G with $V(Q)\cap V(C)=\{\xi_b^+\}$. Since C is extreme, we have $V(Q)\cap V(P)=\emptyset$. Next, since P is a longest path in $G\backslash C$, we have $|Q|\leq\overline{p}+1$. Further, recalling that $\xi_b^+\xi_a,\xi_b^+\xi_{b+1},\xi_b^+\xi_a^+\notin E$ (see Case a1.2), we conclude that $v\xi_a,v\xi_{b+1},v\xi_a^+\notin E$, as well. If $vy\notin E$ for each $y\in(\xi_b^{+2}\overrightarrow{C}\xi_{b+1}^-)$, then clearly

$$N(v)\subseteq(V(Q)\cup\{\xi_1,...,\xi_s\})\backslash\{\xi_a,\xi_{b+1}\xi_a^+\},$$

that is $d(v)\leq|Q|+s-2\leq\overline{p}+s-1=\delta-1$, a contradiction. Now let $vy\in E$ for some $y\in V(\xi_b^{+2}\overrightarrow{C}\xi_{b+1}^-)$. Assume that y is chosen so as to minimize $|\xi_b^+\overrightarrow{C}y|$. Since C is extreme, we have $|\xi_b^+\overrightarrow{C}y|\geq|Q|+1$. Further, since

$$|N(v)\cap V(y\overrightarrow{C}\xi_{b+1}^-)|\geq\delta-(s-2)-|Q|,$$

we have

$$|\xi_b^+\overrightarrow{C}\xi_{b+1}^-|\geq|Q|+1+2(\delta-s+1-|Q|)$$
$$=2\delta-|Q|-2s+3\geq 2\delta-\overline{p}-2s+2=\overline{p}+2.$$

But then $|I_b|\geq\overline{p}+4$, a contradiction.

Case a1.2.2. $N(\xi_b^+)\subseteq V(C)$.

Since $\mu(\Upsilon)=1$ and $\xi_b^+\xi_a^+\notin E$, we have

$$N(\xi_b^+)\subseteq V(\xi_b^{+2}\overrightarrow{C}\xi_{b+1}^-)\cup\{\xi_1,...,\xi_s\}\backslash\{\xi_a,\xi_{b+1}\}.$$

If $\xi_a\neq\xi_{b+1}$, then $d(\xi_b^+)\leq\overline{p}+s-1=\delta-1$, a contradiction. Hence $\xi_a=\xi_{b+1}$.

Case a1.2.2.1. $|I_f|=\overline{p}+2$ for some $f\in\{1,...,s\}\backslash\{a,b\}$.

If $N(\xi_f^+)\subseteq V(C)$, then as indicated above,

$$d(\xi_f^+)\leq s-1+|\xi_f^+\overrightarrow{C}\xi_{f+1}^-|=\overline{p}+s-1=\delta-1,$$

a contradiction. If $N(\xi_f^+)\nsubseteq V(C)$, then we can argue as in Case a1.2.1.

Case a1.2.2.2. $|I_i|\geq\overline{p}+3$ for each $i\in\{1,...,s\}\backslash\{a,b\}$.

If $s\geq 4$, then

$$|C|=\sum_{i=1}^{s}|I_i|\geq s(\overline{p}+3)=(\delta-\overline{p})(\overline{p}+3)$$

$$= 2\delta + 2\overline{p} + 4 + (\delta - \overline{p} - 4)(\overline{p} + 1) \geq 2\delta + 4,$$

contradicting (1). Hence, $s \leq 3$. Moreover, if $s = 3$, then by Claim 1(a5), $|I_1| = |I_2| = |I_3| = \overline{p} + 3$.

Case a2. $z_1z_3, z_2z_3 \in \Upsilon(I_1, ..., I_s)$, where $z_3 \in V(I_f^*)$ and $f \in \{1, ..., s\}\backslash\{a, b\}$.

Assume w.l.o.g. that ξ_a, ξ_b, ξ_f occur on $\overrightarrow{C}$ in a consecutive order. Put

$$|\xi_a\overrightarrow{C}z_1| = d_1, \ |z_1\overrightarrow{C}\xi_{a+1}| = d_2, \ |\xi_b\overrightarrow{C}z_2| = d_3,$$

$$|z_2\overrightarrow{C}\xi_{b+1}| = d_4, \ |\xi_f\overrightarrow{C}z_3| = d_5, \ |z_3\overrightarrow{C}\xi_{f+1}| = d_6.$$

By Claim 2,

$$d_1 + d_3 \geq \overline{p} + 3, \ d_1 + d_5 \geq \overline{p} + 3, \ d_2 + d_4 \geq \overline{p} + 3,$$

$$d_2 + d_6 \geq \overline{p} + 3, \ d_3 + d_5 \geq \overline{p} + 3, \ d_4 + d_6 \geq \overline{p} + 3.$$

Summing up, we get

$$2\sum_{i=1}^{6} d_i = 2(|I_a| + |I_b| + |I_f|) \geq 6(\overline{p} + 3).$$

On the other hand, by Claim 1(a5), $|I_a| + |I_b| + |I_f| \leq 3(\overline{p} + 3)$, implying that $d_1 = d_2 = ... = d_6 = (\overline{p} + 3)/2$ and $\overline{p}$ is odd. Hence $d_i \geq 2$ and using Claim 3, we can state that $G\backslash\{\xi_1, ..., \xi_s, z_1, z_2\}$ has at least $s + 3$ components, contradicting $\tau \geq 1$. Claim 6 is proved.

Claim 7. Either $\mu(\Upsilon) = 1$ or $\mu(\Upsilon) = 3$.

Proof. The proof is by contradiction. If $\mu(\Upsilon) = 0$, then $G\backslash\{\xi_1, ..., \xi_s\}$ has at least $s + 1$ components, contradicting $\tau \geq 1$. Let $\mu(\Upsilon) \geq 1$.

Case a1. $\mu = 2$.

By Claim 5, $\Upsilon(I_1, ..., I_s)$ consists of two crossing intermediate independent edges $w_1w_2 \in \Upsilon(I_f, I_g)$ and $w_3w_4 \in \Upsilon(I_f, I_h)$ for some distinct f, g, h. Assume that both ξ_f, ξ_g, ξ_h and w_1, w_3, w_2, w_4 occur on $\overrightarrow{C}$ in a consecutive order. Put

$$Q = \xi_f\overrightarrow{C}w_1w_2\overrightarrow{C}w_4w_3\overrightarrow{C}\xi_gx_2\overleftarrow{P}x_1\xi_{h+1}\overrightarrow{C}\xi_f,$$

$$|\xi_f\overrightarrow{C}w_1| = d_1, \ |w_1\overrightarrow{C}w_3| = d_2, \ |w_3\overrightarrow{C}\xi_{f+1}| = d_3,$$

$$|\xi_g\overrightarrow{C}w_2| = d_4, \ |w_2\overrightarrow{C}\xi_{g+1}| = d_5, \ |\xi_h\overrightarrow{C}w_4| = d_6, \ |w_4\overrightarrow{C}\xi_{h+1}| = d_7.$$

Clearly, $|Q| = |C| - d_2 - d_4 - d_7 + \overline{p} + 4$. Since $|Q| \leq |C|$, we have $d_2 + d_4 + d_7 \geq \overline{p} + 4$. If $d_3 + d_6 \geq \overline{p} + 3$ and $d_1 + d_5 \geq \overline{p} + 3$, then $\sum_{i=1}^{7} d_i = |I_f| + |I_g| + |I_h| \geq 3\overline{p} + 10$, contradicting Claim 1(a5). Otherwise, either $d_3 + d_6 \leq \overline{p} + 2$ or $d_1 + d_5 \leq \overline{p} + 2$, say $d_3 + d_6 \leq \overline{p} + 2$. Further, if either $d_7 = 1$ or $\xi_{h+1}^-w_3 \in E$, then by Claim 2, $d_3 \geq \overline{p} + 2$, that is $d_3 + d_6 \geq \overline{p} + 3$, a contradiction. Hence, $d_7 \geq 2$ and $\xi_{h+1}^-w_3 \notin E$. By Claim 4, $\xi_{h+1}^-\xi_{f+1}, \xi_{h+1}^-\xi_h \notin E$. If $|I_h| \geq \overline{p} + 4$, then taking into account that $|I_f| + |I_g| \geq 2\overline{p} + 6$ (by Claim 1(a1)), we get $|I_f| + |I_g| + |I_h| \geq 3\overline{p} + 10$, contradicting Claim 1(a5). Hence, $|I_h| \leq \overline{p} + 3$. By a symmetric argument, $|I_g| \leq \overline{p} + 3$.

Case a1.1. $N(\xi_{h+1}^-) \subseteq V(C)$.

If $\xi_{h+1}^{-}w_2 \notin E$, then recalling that $\mu(\Upsilon)=2$, we get

$$N(\xi_{h+1}^{-}) \subseteq V(w_4\overrightarrow{C}\xi_{h+1}^{-2}) \cup \{\xi_1,...,\xi_s\}\backslash\{\xi_{f+1},\xi_h\},$$

implying that $|N(\xi_{h+1}^{-})| \leq \overline{p}+s-1=\delta-1$, a contradiction. Now let $\xi_{h+1}^{-}w_2 \in E$. By Claim 1(a1 and a5), $|I_f|=|I_g|=|I_h|=\overline{p}+3$. Moreover, by Claim 2, $d_5=\overline{p}+2$ and $d_4=1$. Then, for the same reason, $d_1=\overline{p}+2$, implying that $|I_a| \geq \overline{p}+4$, a contradiction.

Case a1.2. $N(\xi_{h+1}^{-}) \not\subseteq V(C)$.
We can argue as in the proof of Claim 6 (Case a1.2.1).

Case a2. $\mu(\Upsilon) \geq 4$.
By Claim 5, there are at least four pairwise crossing intermediate independent edges in $\Upsilon(I_1,...,I_s)$, which is impossible. Claim 7 is proved.

Claim 8. If $\mu(\Upsilon)=1$, then either $n \equiv 1(mod\ 3)$ with $c \geq 2\delta+2$ or $n \equiv 1(mod\ 4)$ with $c \geq 2\delta+3$ or $n \equiv 2(mod\ 3)$ with $c \geq 2\delta+3$.
Proof. By Claim 6, $s \leq 3$ and either $\xi_a^{+}\xi_{b+1}^{-} \in E$ or $\xi_{a+1}^{-}\xi_b^{+} \in E$, say $\xi_{a+1}^{-}\xi_b^{+} \in E$.

Case a1. $s=2$.
It follows that $\delta=\overline{p}+s=\overline{p}+2$. Let $a=1$ and $b=2$. By Claim 2, $|\xi_1\overrightarrow{C}\xi_2^{-}| \geq \overline{p}+2$ and $|\xi_2^{+}\overrightarrow{C}\xi_1| \geq \overline{p}+2$, implying that $|I_i| \geq \overline{p}+3$ $(i=1,2)$.

Case a1.1. $|I_1|=\overline{p}+4$ and $|I_2|=\overline{p}+3$.
If $V(G)=V(C\cup P)$, then $n=3\overline{p}+8=3\delta+2 \equiv 2(mod\ 3)$ with $c=2\overline{p}+7=2\delta+3$, and we are done. Otherwise $N(v_1) \not\subseteq V(C\cup P)$ for some $v_1 \in V(C\cup P)$. Observing that $x_1x_2 \in E$ and recalling that P is a longest path in $V(G\backslash C)$, we conclude that $v_1 \notin V(P)$. Choose a longest path $Q=v_1\overrightarrow{Q}v_2$ with $V(Q)\cap V(C)=\{v_1\}$. Clearly, $1 \leq |Q| \leq \overline{p}+1=\delta-1$ and $N(v_2) \subseteq V(C\cup Q)$.

Case a1.1.1. $v_1 \in V(\xi_2^{+2}\overrightarrow{C}\xi_1^{-})$.
By Claim 1(a6), $N(v_2)\cap V(I_1^{*})=\emptyset$, that is $N(v_2) \subseteq V(I_1)\cup V(Q)$. Assume that v_1 is chosen so as to minimize $|v_1\overrightarrow{C}\xi_1|$, implying that $N(v_2)\cap V(v_1\overrightarrow{C}\xi_1^{-})=\emptyset$. Clearly, $|v_1\overrightarrow{C}\xi_1| \leq \overline{p}+1$. Then by Claim 4, $v_1\xi_2 \notin E$ and therefore, $v_2\xi_2 \notin E$, as well.

Case a1.1.1.1. $v_2\xi_1 \in E$.
It follows that $N(v_2) \subseteq V(Q)\cup V(\xi_2^{+}\overrightarrow{C}v_1^{-})\cup\{\xi_1\}$. Since C is extreme and $v_2\xi_1 \in E$, we have $|v_1\overrightarrow{C}\xi_1| \geq |Q|+1$. If $N(v_2) \subseteq V(Q)\cup\{\xi_1\}$, then clearly $|Q| \geq \delta-1=\overline{p}+1$ and therefore, $|v_1\overrightarrow{C}\xi_1| \geq \overline{p}+2$. But then $|I_2| \geq \overline{p}+4$, a contradiction. Hence, $N(v_2) \not\subseteq V(Q)\cup\{\xi_1\}$, that is $v_2y \in E$ for some $y \in V(\xi_2^{+}\overrightarrow{C}v_1^{-})$. Assume that y is chosen so as to minimize $|y\overrightarrow{C}v_1|$. Observing that $|y\overrightarrow{C}v_1| \geq |Q|+1$ and $\delta=|\xi_2^{+}\overrightarrow{C}\xi_1| \geq 4$, we get

$$|\xi_2^{+}\overrightarrow{C}\xi_1| \geq 2(|Q|+1)+2(\delta-|Q|-2)=2\delta-2 \geq \delta+2=\overline{p}+4,$$

a contradiction.

Case a1.1.1.2. $v_2\xi_1 \notin E$.

It follows that $N(v_2) \subseteq V(Q) \cup V(\xi_2^+ \overrightarrow{C} v_1^-)$. If $N(v_2) \subseteq V(Q)$, then $|Q| \geq \delta = \overline{p}+2$, a contradiction. Otherwise $v_2 y \in E$ for some $y \in V(\xi_2^+ \overrightarrow{C} v_1^-)$. Assume that y is chosen so as to minimize $|y \overrightarrow{C} v_1|$. Since $|y \overrightarrow{C} v_1| \geq |Q|+1$, we have

$$|\xi_2^+ \overrightarrow{C} v_1| \geq |Q| + 1 + 2(\delta - |Q| - 1) = 2\delta - |Q| - 1 \geq \delta = \overline{p} + 2.$$

But then $|I_b| \geq 4$, a contradiction.

Case a1.1.2. $v_1 \in V(\xi_1^+ \overrightarrow{C} \xi_2^{-3})$.

By Claim 1(a6), $N(v_2) \cap V(I_2^*) = \emptyset$, that is $N(v_2) \subseteq V(Q) \cup V(I_1)$. Assume that v_1 is chosen so as to minimize $|\xi_1 \overrightarrow{C} v_1|$, implying that $N(v_2) \cap V(\xi_1^+ \overrightarrow{C} v_1^-) = \emptyset$. Clearly, $|\xi_1 \overrightarrow{C} v_1| \leq \overline{p}+1$. Then by Claim 4, $v_1\xi_2 \notin E$ and therefore, $v_2\xi_2 \notin E$.

Case a1.1.2.1. $\xi_2^+\xi_2^{-2} \in E$.

By Claim 3, $v_1\xi_2^- \notin E$, implying that $v_2\xi_2^- \notin E$.

Case a1.1.2.1.1. $v_2\xi_1 \in E$.

It follows that $N(v_2) \subseteq V(Q) \cup V(v_1 \overrightarrow{C} \xi_2^{-2}) \cup \{\xi_1\}$. Since C is extreme and $v_2\xi_1 \in E$, we have $|\xi_1 \overrightarrow{C} v_1| \geq |Q|+1$. If $N(v_2) \subseteq V(Q) \cup \{\xi_1\}$, then $|Q| \geq \delta - 1 = \overline{p}+1$ and therefore, $|\xi_1 \overrightarrow{C} v_1| \geq \overline{p}+2$. But then $|I_1| \geq \overline{p}+5$, a contradiction. Hence, $N(v_2) \nsubseteq V(Q) \cup \{\xi_1\}$, that is $v_2 y \in E$ for some $y \in V(v_1^+ \overrightarrow{C} \xi_2^{-2})$. Assume that y is chosen so as to minimize $|v_1 \overrightarrow{C} y|$. Observing that $|v_1 \overrightarrow{C} y| \geq |Q|+1$ and $\delta = |\xi_1 \overrightarrow{C} \xi_2^{-2}| \geq 4$, we get

$$|\xi_1 \overrightarrow{C} \xi_2^{-2}| \geq 2(|Q|+1) + 2(\delta - |Q| - 2) = 2\delta - 2 \geq \delta + 2 = \overline{p} + 4,$$

a contradiction.

Case a1.1.2.1.2. $v_2\xi_1 \notin E$.

It follows that $N(v_2) \subseteq V(Q) \cup V(v_1 \overrightarrow{C} \xi_2^{-2})$. If $N(v_2) \subseteq V(Q)$, then $|Q| \geq \delta = \overline{p}+2$, a contradiction. Otherwise $v_2 y \in E$ for some $y \in V(v_1^+ \overrightarrow{C} \xi_2^{-2})$. By choosing y so as to minimize $|v_1 \overrightarrow{C} y|$, we get

$$|v_1 \overrightarrow{C} \xi_2^{-2}| \geq |Q| + 1 + 2(\delta - |Q| - 1) = 2\delta - |Q| - 1 \geq \delta = \overline{p} + 2.$$

This yields $|I_a| \geq \overline{p}+5$, a contradiction.

Case a1.1.2.2. $\xi_2^+\xi_2^{-2} \notin E$.

If $v_2\xi_1 \in E$, then as in Case a1.1.2.1.1, $|\xi_1 \overrightarrow{C} \xi_2^-| \geq \overline{p}+4$, contradicting the fact that $|I_1| = \overline{p}+4$. Otherwise, as in Case a1.1.2.1.2, $|v_1 \overrightarrow{C} \xi_2^-| \geq \overline{p}+2$. Since $|I_1| = \overline{p}+4$, we have $v_1 = \xi_1^+$, $|Q| = \delta - 1 = \overline{p}+1$ and $v_3 = \xi_2^-$. Moreover, we have $N(v_2) = (V(Q) \cup \{\xi_2^-\}) \backslash \{v_2\}$. Further, let v be an arbitrary vertex in $V(Q)\backslash\{v_1\}$. Put $Q' = v_1 \overrightarrow{Q} v^- v_2 \overleftarrow{Q} v$. Since Q' is another longest path with $V(Q') \cap V(C) = \{v_1\}$, we can suppose that $N(v) = (V(Q) \cup \{\xi_2^-\}) \backslash \{v\}$ for each $v \in V(Q)\backslash\{v_1\}$. Furthermore, if $\xi_1 y \in E$ for some $y \in V(\xi_1^{+2} \overrightarrow{C} \xi_2^{-2})$, then

$$\xi_1 x_1 \overrightarrow{P} x_2 \xi_2 \xi_2^+ \xi_2^- v_2 \overleftarrow{Q} v_1 \overrightarrow{C} y \xi_1$$

is longer than C, a contradiction. Hence, $\xi_1 y \notin E$ for each $y \in V(\xi_1^{+2} \overrightarrow{C} \xi_2^{-2})$. Analogously, if $y\xi_2 \in E$ for some $y \in V(\xi_1^{+} \overrightarrow{C} \xi_2^{-2})$, then

$$\xi_1 x_1 \overrightarrow{P} x_2 \xi_2 y \overleftarrow{C} \xi_1^{+} \overrightarrow{Q} v_2 \xi_2^{-} \xi_2^{+} \overrightarrow{C} \xi_1$$

is longer than C, a contradiction. Hence, $y\xi_2 \notin E$ for each $y \in V(\xi_1^{+} \overrightarrow{C} \xi_2^{-2})$. But then $G \backslash \{\xi_1^{+}, \xi_2^{-}\}$ has at least three components, contradicting $\tau \geq 1$.

Case a1.1.3. $v_1 = \xi_2^{-2}$.

By Claim 1(a6), $N(v_2) \subseteq V(I_1)$. If $v_2 y \in E$ for some $y \in V(\xi_1^{+} \overrightarrow{C} v_1^{-})$, then we can argue as in Case a1.1.2. Hence, $N(v_2) \subseteq V(Q) \cup \{\xi_1, \xi_2\}$. If $v_2\xi_2 \in E$, then

$$\xi_1 x_1 \overrightarrow{P} x_2 \xi_2 v_2 \overleftarrow{Q} v_1 \xi_2^{-} \xi_2^{+} \overrightarrow{C} \xi_1$$

is longer than C, a contradiction. Then clearly, $v_2\xi_1 \in E$ and $N(v_2) \subseteq V(Q) \cup \{\xi_1\}$. Furthermore, we have $|Q| \geq \delta - 1$, implying that $|\xi_1 \overrightarrow{C} v_1| \geq |Q| + 1 \geq \delta$. Since $|\xi_1 \overrightarrow{C} v_1| = \delta$, we have $|Q| = \delta - 1 = \overline{p} + 1$ and $N(v_2) = (V(Q) \cup \{\xi_1\}) \backslash \{v_2\}$. Moreover, as in Case 1.1.2.2, we have $N(v) = (V(Q) \cup \{\xi_1\}) \backslash \{v\}$ for each $v \in V(Q) \backslash \{v_1\}$. Now consider an arbitrary vertex $y \in V(\xi_1^{+} \overrightarrow{C} \xi_2^{-3})$. Clearly, $|\xi_1 \overrightarrow{C} y| \leq \overline{p} + 1$. By Claim 2, $y\xi_2^{+} \notin E$. Next, by Claim 4, $y\xi_2 \notin E$. Further, if $y\xi_2^{-} \in E$, then

$$\xi_1 x_1 \overrightarrow{P} \xi_2 \xi_2 \xi_2^{+} \xi_2^{-} y \overrightarrow{C} \xi_2^{-2} \overrightarrow{Q} v_2 \xi_1$$

is longer than C, a contradiction. Finally, since $\mu(\Upsilon) = 1$, we have $yv \notin E$ for each $v \in V(\xi_2^{+2} \overrightarrow{C} \xi_1^{-})$. But then $G \backslash \{\xi_1, \xi_2^{-2}\}$ has at least three components, contradicting $\tau \geq 1$.

Case a1.1.4. $v_1 = \xi_1$.

If $v_2 v_3 \in E$ for some $v_3 \in V(\xi_2^{+2} \overrightarrow{C} \xi_1^{-}) \cup V(\xi_1^{+} \overrightarrow{C} \xi_2^{-2})$, then we can argue as in Cases a1.1.1-a1.1.3. Otherwise $v_2 v_3 \in E$ for some $v_3 \in \{\xi_2^{-}, \xi_2^{+}, \xi_2\}$. If $v_3 \in \{\xi_2, \xi_2^{+}\}$, then we can show, as in Case a1.1.3, that $G \backslash \{\xi_1, v_3\}$ has at least three components, contradicting $\tau \geq 1$. Now let $v_3 = \xi_2^{-}$. Consider an arbitrary vertex $v \in V(Q) \backslash \{v_1\}$. Since C is extreme, we have $N(v) \cap \{\xi_2, \xi_2^{+}\} = \emptyset$. Next, if $vy \in E$ for some $y \in V(C) \backslash \{\xi_1, \xi_2, \xi_2^{-}, \xi_2^{+}\}$, then we can argue as in Cases a1.1.1-a1.1.3. Thus, we can assume that $N(v) \subseteq V(Q) \cup \{\xi_2^{-}\}$, implying that $|Q| \geq \delta - 1 = \overline{p} + 1$. Let $w \in V(\xi_1^{+} \overrightarrow{C} \xi_2^{-3})$. Since $|\xi_1 \overrightarrow{C} w| \leq \overline{p} + 1$, we have $w\xi_2^{+} \notin E$ (by Claim 2) and $w\xi_2 \notin E$ (by Claim 4). Recalling also that $\mu(\Upsilon) = 1$, we conclude that $N(v) \subseteq V(\xi_1 \overrightarrow{C} \xi_2^{-})$. If $\xi_2^{-2}\xi_2, \xi_2^{-2}\xi_2^{+} \notin E$, then clearly $G \backslash \{\xi_1, \xi_2^{-}\}$ has at least three components, contradicting $\tau \geq 1$. Hence, either $\xi_2^{-2}\xi_2 \in E$ or $\xi_2^{-2}\xi_2^{+} \in E$.

Case a1.1.4.1. $\xi_2^{-2}\xi_2 \in E$.

If $\xi_2^{-2}\xi_2^{+} \notin E$, then $G \backslash \{\xi_1, \xi_2, \xi_2^{-}\}$ has at least four components, contradicting $\tau \geq 1$. Hence, $\xi_2^{-2}\xi_2^{+} \in E$, that is $\langle \xi_2, \xi_2^{-}, \xi_2^{-2}, \xi_2^{+} \rangle$ is a complete graph. If $V(G) = V(C \cup P \cup Q)$, then $n = 4\delta + 1 \equiv 1 (mod\ 4)$ with $c = 2\delta + 3$, and we are done. Otherwise, as in previous cases, we can show that $\tau < 1$, a contradiction.

Case a1.1.4.2. $\xi_2^{-2}\xi_2^{+} \in E$.

If $\xi_2^{-2}\xi_2 \notin E$, then $G \backslash \{\xi_1, \xi_2^{-}, \xi_2^{+}\}$ has at least four components, contradicting $\tau \geq 1$. Otherwise $\langle \xi_2, \xi_2^{-}, \xi_2^{-2}, \xi_2^{+} \rangle$ is a complete graph and we can argue as in Case a1.1.4.1.

Case a1.1.5. $v_1 \in \{\xi_2, \xi_2^-, \xi_2^+\}$.

Since C is extreme, we have $v_2 \notin \{\xi_2, \xi_2^-, \xi_2^+\}$ and therefore, we can argue as in Cases a1.1.1-1.1.4.

Case a1.2. $|I_1| = |I_2| = \overline{p} + 3$.

We can show that $n = 3\delta + 1 \equiv 1(mod\ 3)$ with $c = 2\delta + 2$, by arguing as in Case a1.1.

Case a2. $s = 3$.

By Claim 6, $|I_1| = |I_2| = |I_3| = \overline{p} + 3 = \delta$ and $\xi_2^- \xi_2^+ \in E$. If $\delta \geq 4$, then $c = 3\delta \geq 2\delta + 4$, contradicting (1). Hence $\delta = 3$ and therefore, $\overline{p} = 0$. Put

$$C = \xi_1 w_1 w_2 \xi_2 w_3 w_4 \xi_3 w_5 w_6 \xi_1,$$

where $w_2 w_3 \in E$. Using Claims 2-5, we can show that

$$N_C(w_1) = \{w_2, \xi_1, \xi_3\},\ N_C(w_6) = \{w_5, \xi_1, \xi_3\}.$$

Analogous relations hold for w_4, w_5. If $V(G\backslash C) = \{x_1\}$, then $n = 10 \equiv 1(mod\ 3)$ with $c = 9 = 2\delta + 3 > 2\delta + 2$, and we are done. Otherwise $N(y) = \{v_1, v_2, v_3\}$ for some $y \in V(G\backslash C)\backslash\{x_1\}$ with $N(y) \subseteq V(C)$. Since C is extreme, it is not hard to see that either $N(y) = \{w_2, \xi_1, \xi_3\}$ or $N(y) = \{w_3, \xi_1, \xi_3\}$ or $N(y) = \{\xi_1, \xi_2, \xi_3\}$. But then $G\backslash N(y)$ has at least four components, contradicting $\tau \geq 1$. Claim 8 is proved.

Claim 9. If $\mu = 3$, then G is the Petersen graph, that is $n = 10 \equiv 1(mod\ 3)$ with $c \geq 2\delta + 2$.

Proof. By Claim 5, $\Upsilon(I_1, ..., I_s)$ contains three pairwise crossing intermediate independent edges e_1, e_2, e_3. Let $e_1 = w_1 w_2$, $e_2 = w_3 w_4$ and $e_3 = w_5 w_6$. If $w_1, w_3, w_5 \in V(I_f^*)$ for some $f \in \{1, ..., s\}$, then we can argue as in proof of Claim 7. Otherwise we can assume w.l.o.g. that $w_1, w_3 \in V(I_f^*)$, $w_2, w_5 \in V(I_g^*)$ and $w_4, w_6 \in V(I_h^*)$ for some distinct $f, g, h \in \{1, ..., s\}$, where both ξ_f, ξ_g, ξ_h and $w_1, w_3, w_5, w_2, w_4, w_6$ occur on $\overrightarrow{C}$ in a consecutive order. By Claim 1(a1 and a5), $|I_f| = |I_g| = |I_h| = \overline{p} + 3$ and $|I_i| = \overline{p} + 2$ for each $i \in \{1, ..., s\}\backslash\{f, g, h\}$. Put

$$|\xi_f \overrightarrow{C} w_1| = d_1,\ |w_1 \overrightarrow{C} w_3| = d_2,\ |w_3 \overrightarrow{C} \xi_{f+1}| = d_3,$$

$$|\xi_g \overrightarrow{C} w_5| = d_4,\ |w_5 \overrightarrow{C} w_2| = d_5,\ |w_2 \overrightarrow{C} \xi_{g+1}| = d_6,$$

$$|\xi_h \overrightarrow{C} w_4| = d_7,\ |w_4 \overrightarrow{C} w_6| = d_8,\ |w_6 \overrightarrow{C} \xi_{h+1}| = d_9.$$

If $d_3 + d_7 \geq \overline{p} + 3$, $d_1 + d_6 \geq \overline{p} + 3$ and $d_4 + d_9 \geq \overline{p} + 3$, then clearly $|I_f| + |I_g| + |I_h| \geq 3\overline{p} + 12$, a contradiction. Otherwise we can assume w.l.o.g. that $d_3 + d_7 \leq \overline{p} + 2$. Further, if either $d_1 \geq 2$ or $d_9 \geq 2$, then we can argue as in the proof of Claim 7 (Case a1.1). Hence, we can assume that $d_1 = d_9 = 1$. By Claim 2, $d_4 = d_6 = 1$. For the same reason, using the fact that $d_1 = d_6 = 1$, we get $d_3 = d_7 = 1$.

Case a1. Either $\xi_{h+1} \neq \xi_f$ or $\xi_{f+1} \neq \xi_g$ or $\xi_{g+1} \neq \xi_h$.

Assume w.l.o.g. that $\xi_{h+1} \neq \xi_f$, implying that $|I_{f-1}| = \overline{p} + 2$. By Claim 5, $\xi_f^- y \notin E$ for each $y \in V(I_i^*)$ and $i \in \{1, ..., s\}\backslash\{f - 1\}$. Moreover, by Claim 4, $\xi_f^- y \notin E$ for each $y \in \{\xi_{f+1}, \xi_h\}$. If $N(\xi_f^-) \subseteq V(C)$, then $d(\xi_f^-) \leq \delta - 1$, a contradiction. Otherwise we can argue as in the proof of Claim 6 (Case a1.2.1).

Case a2. $\xi_{h+1}=\xi_f$, $\xi_{f+1}=\xi_g$, $\xi_{g+1}=\xi_h$.
It follows that $s=3$. Assume w.l.o.g. that $f=1$, $g=2$ and $h=3$.

Case a2.1. Either $d_2\geq 2$ or $d_5\geq 2$ or $d_8\geq 2$.
Assume w.l.o.g. that $d_2\geq 2$, that is $w_1^+\neq w_3$. If $\overline{p}=0$, then $|I_1|=3$, implying that $d_2=1$, a contradiction. Let $\overline{p}\geq 1$. By Claim 4, $w_1^+\xi_2, w_1^+\xi_3\notin E$. If $N(w_1^+)\subseteq V(C)$, then by Claim 4, $N(w_1^+)\subseteq V(w_1^{+2}\overrightarrow{C}w_3)\cup\{\xi_1\}$. Since $|I_1|=\overline{p}+3$, we have $|w_1^+\overrightarrow{C}w_3|\leq\overline{p}$. But then $d(w_1^+)\leq\overline{p}+1=\delta-2$, a contradiction. If $N(w_1^+)\not\subseteq V(C)$, then we can argue as in the proof of Claim 6 (Case a1.2.1).

Case a2.2. $d_2=d_5=d_8=1$.
It follows that $|I_i|=3$ $(i=1,2,3)$, that is $\overline{p}=0$, $\delta=3$ and $c=9$. Clearly $\langle V(C)\cup\{x_1\}\rangle$ is the Petersen graph. If $V(G\backslash C)\neq\{x_1\}$, then it is not hard to see that $c\geq 10$, a contradiction. Otherwise, $n=10\equiv 1(mod\ 3)$ with $c=9=2\delta+3>2\delta+2$. Claim 9 is proved.

Thus, the result holds from Claims 7,8,9.

Case 2. $\overline{p}=\delta-1$.
Clearly, $|N_C(x_i)|\geq 1$ $(i=1,2)$.

Case 2.1. $x_1y_1, x_2y_2\in E$ for some distinct $y_1,y_2\in V(C)$.
We distinguish three main subcases.

Case 2.1.1. There exists a path $Q=z\overrightarrow{Q}y$ with $z\in V(P)$, $y\in V(C)\backslash\{y_1,y_2\}$ and $V(Q)\cap V(C\cup P)=\{z,y\}$.
Assume w.l.o.g. that $y\in V(y_1^+\overrightarrow{C}y_2^-)$. Since C is extreme, we have

$$|y_1\overrightarrow{C}y|\geq|x_1\overrightarrow{P}z|+2,\ |y\overrightarrow{C}y_2|\geq|z\overrightarrow{P}x_2|+2,\ |y_2\overrightarrow{C}y_1|\geq\delta+1.$$

Summing up, we get $|C|\geq 2\delta+4$, contradicting (1).

Case 2.1.2. There exists a path $Q=z\overrightarrow{Q}y$ with $z\in V(y_1^+\overrightarrow{C}y_2^-)$, $y\in V(y_2^+\overrightarrow{C}y_1^-)$ and $V(Q)\cap V(C\cup P)=\{z,y\}$.
By Claim 1(a1), $|C|\geq 2\overline{p}+6=2\delta+4$, contradicting (1).

Case 2.1.3. $G\backslash\{y_1,y_2\}$ has at least three components.
It follows that $\tau<1$, contradicting the hypothesis.

Case 2.2. $N_C(x_1)=N_C(x_2)=\{y\}$ for some $y\in V(C)$.
It follows that

$$N(x_1)=(V(P)\cup\{y\})\backslash\{x_1\},\ N(x_2)=(V(P)\cup\{y\})\backslash\{x_2\}.$$

Moreover, $x_1\overrightarrow{P}v^-x_2\overleftarrow{P}v$ is a longest path in $G\backslash C$ for each $v\in V(x_1^+\overrightarrow{P}x_2)$. Since G is 2-connected, we have $wz\in E$ for some $w\in V(P)$ and $z\in V(C)\backslash\{y\}$. If $w=x_1$, then using the path $zx_1\overrightarrow{P}x_2y$, we can argue as in Case 2.1. Otherwise we can use the path $yx_1\overrightarrow{P}w^-x_2\overleftarrow{P}wz$.

Case 3. $\overline{p} \geq \delta$.

Case 3.1. $x_1y_1, x_2y_2 \in E$ for some distinct $y_1, y_2 \in V(C)$.

Clearly, $|y_1\overrightarrow{C}y_2| \geq \delta+2$ and $|y_2\overrightarrow{C}y_1| \geq \delta+2$, which yields $|C| \geq 2\delta+4$, contradicting (1).

Case 3.2. $N_C(x_1) = N_C(x_2) = \{y\}$ for some $y \in V(C)$.

Let $y_1, y_2, ..., y_t$ be the elements of $N_P^+(x_2)$ occurring on $\overrightarrow{P}$ in a consecutive order. Put $H = \langle V(y_1^-\overrightarrow{P}x_2)\rangle$ and

$$P_i = x_1\overrightarrow{P}y_i^-x_2\overleftarrow{P}y_i \ (i = 1, ..., t).$$

Since P_i is a longest path in $G\backslash C$ for each $i \in \{1, ..., t\}$, we can assume w.l.o.g. that P is chosen so as to maximize $|V(H)|$. If $y_iz \in E$ for some $i \in \{1, ..., t\}$ and $z \in V(C)\backslash\{y\}$, then we can argue as in Case 3.1. Otherwise $N(y_i) \subseteq V(H) \cup \{y\}$ $(i = 1, ..., t)$, that is $|N_H(y_i)| \geq \delta - 1$ $(i = 1, ..., t)$. By Lemma 3, for each distinct $u, v \in V(H)$, there is a path in H of length at least $\delta - 1$, connecting u and v. Since G is 2-connected, H and C are connected by two vertex disjoint paths. This means that there is a path $Q = y_1\overrightarrow{Q}y_2$ of length at least $\delta + 1$ with $V(Q) \cap V(C) = \{y_1, y_2\}$. Further, we can argue as in Case 2.1.

Case 3.3. Either $N_C(x_1) = \emptyset$ or $N_C(x_2) = \emptyset$.

Assume w.l.o.g. that $N_C(x_1) = \emptyset$. By arguing as in Case 3.2, we can find a path $Q = y_1\overrightarrow{Q}y_2$ of length at least $\delta + 2$ with $V(Q) \cap V(C) = \{y_1, y_2\}$, and the result follows immediately. Theorem 1 is proved. ■

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Բաուերի և Շմայխելի թեորեմի լավացում

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Ամփոփում

Դիցուք G-ն n գագաթ և δ նվազագույն աստիճան ունեցող գրաֆ է: Գրաֆի ամենաերկար ցիկլի c երկարության առաջին ոչ պարզունակ գնահատականը ստացել է Դիրակը (1952). (i) Կամայական 2-կապակցված գրաֆում, $c \geq \min\{n, 2\delta\}$: Այս արդյունքը 1986թ-ին Բաուերը և Շմայխելը լավացրեցին 1-կոշտ գրաֆների համար. (ii) Կամայական 1-կոշտ գրաֆում, $c \geq \min\{n, 2\delta+2\}$: Ստացված երկու գնահատականներն էլ հասանելի են n պարամետրի որոշակի արժեքների համար: Ներկա աշխատանքում բերվում է Բաուերի և Շմայխելի գնահատականի մի լավացում, որը հասանելի է n պարամետրի ցանկացած արժեքի դեպքում:

Улучшение Теоремы Бауера и Шмейхеля

Ж. Никогосяан

Аннотация

Пусть G является n вершинным графом с минимальной степенью δ. В 1952г. Дирак получил первую нетривиальную оценку для длины c длиннейшего цикла графа G: (i) В любом 2-связном графе, $c \geq \min\{n, 2\delta\}$. Эту оценку в 1986г. Бауер и Шмейхель улучшили для 1-жестких графов: (ii) В любом 1-жестком графе, $c \geq \min\{n, 2\delta + 2\}$. Полученные оценки достигаемы для определенных значений параметра n. В настоящей работе предлагается улучшение оценки Бауера и Шмейхелья, которое неулучшаема для всех значений параметра n.