Mathematical Problems of Computer Science 39, 94–100, 2013.

Radicals and Preradicals in the Category of Modules over All Rings

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Abstract

Let Mod be a category whose objects are all possible pairs (A, U), where U is an associative ring, A is a right U-module (not unitary in the general case) and the set of morphisms of module (A, U) to module (B, V) consists of pairs of mappings (φ_A, φ_U), where φ_A or φ_U, respectively, is a homomorphism of Abelian group A to Abelian group B (ring U to ring V), where $(a \cdot u)\varphi_A = a\ \varphi_A \cdot u\ \varphi_U$, $a \in A$, $u \in U$. This pair of mappings is called a homomorphism of module (A, U) to module (B, V). It is proved that strict radicals of Mod in the sense of Kurosh are described by means of systems of strict radicals of the category of associative rings As and categories of right U-modules $Mod - U$, $U \in As$. It turned out that a wider classes of preradicals and radicals of Mod can also be described by means of systems of preradicals and radicals of As and $Mod - U$, $U \in As$, respectively, which satisfy some conditions.

Keywords: Module, associative ring, Abelian group, category, radical, preradical, torsion, ideally hereditary radical, strongly hereditary radical.

Let us consider the category Mod, whose objects are all possible pairs (A, U), where A is a right U-module over an associative ring U, and the morphisms are pairs (φ_A, φ_U) of homomorphisms satisfying the condition $(a \cdot u)\ \varphi_A = a\ \varphi_A \cdot\ u\ \varphi_U$, $a \in A$, $u \in U$ [1].

Definition 1: *We will say that a preradical r is defined in the Mod, if an ideal $r(A, U)$ matches to each module $(A, U) \in Mod$ such that $r(A, U)(\varphi_A, \varphi_U) \subseteq r(B, V)$ for each homomorphism $(\varphi_A, \varphi_U) : (A, U) \to (B, V)$ of Mod [2], [3].*

In other words, the preradical r is a normal subfunctor of the identity functor Id_{Mod} : $Mod \to Mod$. In each module $(A, U) \in Mod$ it singles out an ideal $r(A, U)$ such that for every homomorphism $(\varphi_A, \varphi_U) : (A, U) \to (B, V)$ the diagram:

$$\begin{array}{ccc} (A, U) & \longrightarrow & (B, V) \\ \uparrow & & \uparrow \\ r(A, U) & \longrightarrow & r(B, V) \end{array}$$

is commutative.

Assume that in the category Mod some preradical r is specified. Consider a full subcategory $Mod(As)$ of Mod whose objects are all modules of the form $(0, U)$. Any ideals and any homomorphic images of objects from $Mod(As)$ themselves lie in subcategory $Mod(As)$, and hence, the preradical r induces a completely determinate preradical R in subcategory $Mod(As)$. This means that a preradical r defines a completely determinate preradical R in category As of associative ring and $r(0,U) = (0, R(U))$.

Lemma 1: *Let r be any preradical in Mod, (A,U) be any module and $r(A,U) = (A', U')$. Then $U' = R(U)$, where R is a preradical in As induced of preradical r of Mod.*

Proof. Since r is a preradical and $(0,U)$ is a submodule of (A,U) then $(0, R(U)) = r(0,U) \subseteq r(A,U) = (A', U')$. Hence $R(U) \subseteq U'$. But for homomorphism $(0, 1_U) : (A,U) \to (0,U)$ we have $(A', U')(0, 1_U) = r(A,U)(0, 1_U) \subseteq r(0,U) = (0, R(U))$, i.e., $U' \subseteq R(U)$. Lemma 1: is proved.

Recall that the submodule (A', U') of (A,U) will be an ideal of (A,U) if and only if U' is an ideal of the ring U and the inclusions $A \cdot U' \subseteq A'$ and $A' \cdot U \subseteq A'$ are observed [1].

Assume r is any preradical of Mod, (A,U) is an any module and $r(A,U) = (A', U')$. Since (A', U') is an ideal of (A,U), then A' is a U-submodule of U-module A.

Now show that matching submodule A' to each U-module A defines the preradical R_u in the category of right U-modules $Mod - U$.

Assume A and B are U-modules, $\varphi : A \to B$ is an U-homomorphism and $r(A,U) = (A', U')$, $r(B,U) = (B', U')$, where $U' = R(U)$ by Lemma 1:. We'll show that $A'\varphi \subseteq B'$. Indeed, since $(\varphi, 1_U) : (A,U) \to (B,U)$ is a homomorphism in Mod, then $(A', U')(\varphi, 1_U) = r(A,U)(\varphi, 1_U) \subseteq r(B,U) = (B', U')$. Hence $A'\varphi \subseteq B'$. Thus, for any U-module A, if $r(A,U) = (A', U')$, matching of U-submodule $R_u(A) = A'$ to each U-module A defines the preradical R_u in the category $Mod - U$. Thus, we have the following lemma.

Lemma 2: *Each preradical r of the category Mod induces a completely determinate preradical R in As and preradicals R_U in the categories $Mod - U$ $(U \in As)$ such that $r(A,U) = (R_U(A), R(U))$ for any module (A,U) of the category Mod.*

Let us consider the action of the ring homomorphism $\varphi : U \to V$. Any right V-module B becomes a right U-module if the action of the operators is defined as follows: $a \circ u = a(u\varphi)$, we will say that the module B is converted into the U-module B_φ by withdrawal along φ.

Assume R is any preradical of As and R_U $(U \in As)$ are preradicals of the categories $Mod - U$. Consider the system of preradicals $\{R; R_U \mid U \in As\}$.

Definition 2: *The system of preradicals $\{R; R_U \mid U \in As\}$ will be called a matched system of preradicals, if it satisfies the following conditions:*

(P1) $A \cdot R(U) \subseteq R_U(A)$ *for each U-module A*

(P2) $R_U(B_\varphi) \subseteq R_V(B))$ *for each V-module B and each homomorphism $\varphi : U \to V$ of rings, where B_φ is the conversion of V-module B into an U-module by the withdrawal along φ.*

Theorem 1: *Assume that the preradical r is specified in Mod. Then in As it induces the preradical R and in each category $Mod - U$ $(U \in As)$ it induces preradicals R_U such that the system of preradicals $\{R; R_U \mid U \in As\}$ is matched. Conversely, every matched system of preradicals $\{R; R_U \mid U \in As\}$ specifies a completely determinate preradical r in Mod such that $r(A,U) = (R_U(A), R(U))$ for each $(A,U) \in Mod$. There exists a one-to-one correspondence between all the preradicals of Mod and all the matched systems of preradicals.*

Proof. Assume that r is a preradical of Mod. Then, by Lemma 2:, we have a completely determinate system of preradicals $\{R; R_U \mid U \in As\}$. Since for each $(A,U) \in Mod$, $r(A,U) = (R_U(A), R(U))$ is an ideal of module (A,U), hence $A \cdot R(U) \subseteq R_U(A)$. Thus, the condition (P1) is satisfied.

Assume that B is a V-module, $\varphi : U \to V$ is a homomorphism of rings, and B_φ is the conversion of V-module B into an U-module by withdrawal along φ. As the pair $(1,\varphi) : (B_\varphi, U) \to (B,V)$ is a homomorphism in Mod, then $(R_U(B_\varphi), R(U))(1,\varphi) = r(B_\varphi, U)(1,\varphi) \subseteq r(B,V) = (R_V(B), R(V))$.

Conversely, assume that we are given a matched system of preradicals $\{R; R_U \mid U \in As\}$ and $(A,U) \in Mod$. Since $R(U)$ is an ideal of ring U, $R_U(A)$ is a submodule of U-module A and $A \cdot R(U) \subseteq R_U(A)$ by condition (P1), then the submodule $(R_U(A), R(U))$ is an ideal of the module (A,U).

Let $(\varphi,\psi) : (A,U) \to (B,V)$ be any homomorphism. It is clear that $R(U)\psi \subseteq R(V)$. On the other hand, it is easily seen that the homomorphism $(\varphi,\psi) = (\varphi,1)\cdot(1,\psi)$, where $(\varphi,1) : (A,U) \to (B_\psi, U)$ and $(1,\psi) : (B_\psi, U) \to (B,V)$. Since R_U is a preradical in $Mod{-}U$ and $(\varphi,1)$ is an U-homomorphism, then $R_U(A)\varphi \subseteq R_U(B_\psi)$. And from the condition (P2) it follows that $R_U(B_\psi) \subseteq R_V(B)$. Hence $R_U(A)\varphi \subseteq R_V(B)$.

The third assertion of this theorem follows from the first two ones, since the expressions given in the theorem above are mutually inversive. Thus, the proof of the theorem is completed.

Remark 1 It is easy to verify [2] that if r is a preradical in Mod, then $r(A,U)$ includes all the r-submodules of the module (A,U) for each module (A,U). Also the class of r-radical modules is enclosed under the operation of homomorphic images.

Definition 3: *A preradical r of Mod is called a radical if $r((A,U)/r(A,U)) = (0,0)$ for each module (A,U) [4].*

Similar to Lemmas 1: and 2:, one may prove the following lemma.

Lemma 3: *Let r be a radical in Mod, (A,U) be some module and let $r(A,U) = (A',U')$. Then $U' = R(U)$, where R is the radical in As induced of the radical r of Mod.*

Lemma 4: *Every radical r of the category Mod induces a completely determinate radical R in As and radicals R_u in categories $Mod - U$ $(U \in As)$ such that $r(A,U) = (R_U(A), R(U))$ for any module (A,U) of the category Mod.*

Assume that R is a radical of As and R_U $(U \in As)$ are radicals of categories $Mod - U$. Consider the system of radicals $\{R, R_U \mid U \in As\}$.

Definition 4: *The system of radicals $\{R, R_U \mid U \in As\}$ is called a matched system of radicals if it satisfies the following conditions:*

(P1) $A \cdot R(U) \subseteq R_U(A)$ *for each U-module A*

(P2) $R_U(B_\varphi) \subseteq R_V(B))$ *for each V-module B and each homomorphism $\varphi : U \to V$ of rings, where B_φ is the conversion of V-module B into an U-module by the withdrawal along φ.*

(P3) $R_{U/R(U)}(A) = R_U(A_\pi))$ *for each* $U/R(U)$*-module* A*, where* $\pi : U \to U/R(U)$ *is a natural epimorphism and* A_π *is the conversion of* $U/R(U)$ *module* A *into* U*-module by withdrawal along* π*.*

Theorem 2: *Assume that some radical* r *is specified in* Mod*. Then it induces the radical* R *in* As *and radicals* R_U *in each category* $Mod - U$*,* $U \in As$ *such that the system of radicals* $\{R, R_U \mid U \in As\}$ *is matched. Conversely, every matched system of radicals* $\{R, R_U \mid U \in As\}$ *specifies a completely determinate radical* r *in* Mod *such that* $r(A, U) = (R_U(A), R(U))$ *for each* $(A, U) \in Mod$*. There exists a one-to-one correspondence between all the radicals of* Mod *and all the matched systems of radicals.*

Proof. Assume that r is a radical in Mod and $\{R, R_U \mid U \in As\}$ is a completely determined system of preradicals. Obviously, if r is a radical in Mod, R is a radical in As. Moreover, $(R_{U/R(U)}(A/R_U(A)), R(U/R(U))) = r((A, U)/r(A, U)) = (0, 0)$, i.e. $R_{U/R(U)}(A/R_U(A)) = 0$ for each $(A, U) \in Mod$. Hence, as it follows from condition (P2) $R_U(A/R_U(A)) = 0$ for every U-module A.

Let $\pi : U \to U/R(U)$ be a natural epimorphism and A an arbitrary $U/R(U)$-module. Consider U-module A_π in which $a \circ u = a * (u + R(U))$, $a \in A$, $u \in U$ by definition. Obviously, every submodule of U-module A_π is a submodule of $U/R(U)$-module A. Since $A \circ R(U) = 0$, the reverse conversion, i.e. conversion to operation $a * (u + R(U)) = a \circ u$, $a \in A$, $u \in U$ also holds. Therefore U-module A_π is transformed into a $U/R(U)$-module A, and the submodules and factormodules of U-module A_π are transformed into submodules and factormodules of $U/R(U)$-module A. Consider a natural U-epimorphism $\theta : A_\pi \to A_\pi/R_U(A_\pi)$. Obviously, for transfer to $*$ operation the mapping θ becomes an $U/R(U)$-homomorphism. Thus, because $R_{U/R(U)}$ is a preradical of $Mod - U/RU$ we have $R_{U/R(U)}(A)\theta \subseteq R_{U/R(U)}(A_\pi/R_U(A_\pi))$. On the other hand, as r is a radical of Mod, then in view of assertion above $R_{U/R(U)}(A_\pi/R_U(A_\pi)) = 0$. Hence $R_{U/R(U)}(A)\theta = 0$, i.e. $R_{U/R(U)}(A) \subseteq R_U(A_\pi)$ for each $U/R(U)$-module A. Therefore, by condition (P2) $R_U(A_\pi) \subseteq R_{U/R(U)}(A)$ for each $U/R(U)$-module A.

Conversely, let (A, U) be a random module of Mod. Since R is a radical of As, then $r((A, U)/r(A, U)) = (R_{U/R(U)}(A/R_U(A), 0))$. An application of condition (P3) to $U/R(U)$-module $A/R_U(A)$ yields $R_{U/R(U)}(A/R_U(A)) = R_U(A/R_U(A)) = 0$, as R_U is a radical of $Mod - U$. The third assertion in this theorem follows from Theorem 1:. The theorem is proved.

Definition 5: *A preradical* r *of* Mod *is called idempotent, if* $r(r(A, U)) = r(A, U)$ *for each module* (A, U)*.*

Theorem 3: *The matched system of preradicals* $\{R; R_U \mid U \in As\}$ *defines an idempotent preradical of the category* Mod *if and only if all preradicals of that system are idempotent and the condition*

(P4) $\quad R_U(A) = R_{R(U)}(A)$

holds for each U*-module* A*.*

Proof. Assume that r is an idempotent preradical of Mod and $\{R; R_U \mid U \in As\}$ is the corresponding matched system of preradicals. Because $r(0, U) = (0, R(U))$ for any $(0, U) \in Mod(As)$ and r is idempotent, R will be also idempotent. Therefore, the idempotence of r

means that $(R_U(A), R(U)) = (R_{R(U)}\ (R_U(A)), R(U))$, i.e. $R_U(A) = R_{R(U)}(R_U(A)$ for each module (A, U) of Mod.

On the other hand, it follows from $(P2)$ that $R_{R(U)}(R_U(A)) \subseteq R_U(R_U(A))$. Hence $R_U(A) = R_{R(U)}(R_U(A)) \subseteq R_U(R_U(A)) \subseteq R_U(A)$, i.e. $R_U(R_U(A)) = R_U(A)$ for each U-module A.

As we have already seen if r is idempotent, $R_U(A) = R_{R(U)}(R_U(A))$ for each U-module A. On the other hand for each U-module A, $R_U(A)$ is a $R(U)$-submodule of $R(U)$-module A, because $R_U(A) \cdot R(U) \subseteq A \cdot R(U) \subseteq R_U(A)$ by condition $(P1)$. Hence $R_{R(U)}(R_U(A)) \subseteq R_{R(U)}(A)$. We obtain that $R_U(A) \subseteq R_{R(U)}(A)$. But in view of $(P2)$ $R_{R(U)}(A) \subseteq R_U(A)$. Hence $R_U(A) = R_{R(U)}(A)$ for each U-module A.

Conversely, assume that $\{R; R_U \mid U \in As\}$ is a matched system of idempotent preradicals which satisfy the condition $(P4)$ and (A, U) is a module of the category Mod. From $(P4)$, which is used for U-module $R_U(A)$, it follows that $R_{R(U)}(R_U(A)) = R_U(R_U(A)) = R_U(A)$ because R_U is an idempotent preradical in Mod-U. Thus, the proof of the theorem is completed as $R(R(U)) = R(U)$ for each $U \in As$.

Definition 6: *The radical r of the category Mod is called ideally hereditary if $r(A', U') = r(A, U) \cap (A', U')$ for all ideals (A', U') of $(A, U) \in Mod$ [2], [5].*

Remark 2 The radical R_0 of the system $\{R; R_U \mid U \in As\}$ is a radical of the category of Abelian group Ab. Since $R_0(A)\eta \subseteq R_0(A)$ for every Abelian group A and every endomorphism η of A, the subgroup $R_0(A)$ will be an U-submodule for each U-module A. Thus, it is easy to prove that R_0 is a radical in all the categories $Mod - U$ $(U \in As)$. The system of radicals $\{R; R_U \mid U \in As\}$in which $R_U = R_0$ for all $U \in As$ turns to a pair of $\{R; R_0\}$. As in such systems the conditions (P2) and (P3) are executed automatically, then they will be matched if they satisfy the only condition: $A \cdot R(U) \subseteq R_0(A)$ for each U-module A, i.e. the only condition (P1).

Theorem 4: *Assume that r is an ideally hereditary radical in Mod. Then in As it induces an ideally hereditary radical R and in Ab it induces a torsion R_{Ab} such that the pair $\{R, R_{Ab}\}$ of radicals is matched. Conversely, every matched pair of radicals $\{R, R_{Ab}\}$, where R is an ideally hereditary radical in As and R_{Ab} is a torsion in Ab, specifies a completely determinate ideally hereditary radical r in Mod whose radical class includes all modules (A, U) such that $U = R(U)$ and $A = R_{Ab}(A)$. There exists a one-to-one correspondence between all ideally hereditary radicals of Mod and all such matched pairs.*

Proof. Let r be an ideally hereditary radical in Mod and let $\{R; R_U \mid U \in As\}$ be its matched system of radicals. We will prove that this matched system, which in particular, satisfies the condition (P1) will be a pair of $\{R; R_0\}$, where R is an ideally hereditary radical in As and R_0 is a torsion in Ab.

Let U be an associative ring and A be any U-module. Consider the module $(A, 0)$. It is easy to prove that $(A, 0)$ is an ideal of module (A, U). Therefore, as r is an ideally hereditary radical in Mod, $(R_0(A), 0) = r(A, 0) = (A, 0) \bigcap r(A, U) = (A \bigcap R_U(A), 0) = (R_U(A), 0)$, i.e. $R_U(A) = R_0(A)$ for each U-module A, where $U \in As$. Thus $R_U = R_0$ for each $U \in As$.

Now let U be an associative ring and V be any ideal of U. It is obvious that the module $(0, V)$ is an ideal of the module $(0, U)$. Therefore, as r is an ideally hereditary radical in Mod, $(0, R(V)) = r(0, V) = (0, V) \bigcap r\,(0, U)) = (0, V \bigcap R(U))$. Hence $R(V) = V \bigcap R(U)$, i.e. R is an ideally hereditary radical in As.

Similarly, considering the modules $(A, 0)$, we can prove that R_0 is a torsion in *Ab*.

Conversely, let the pair of radicals $\{R; R_0\}$ satisfy the condition of the theorem. As this pair satisfies condition (P1), it will be a matched system of radicals according to the remark above. Therefore, the considered system will define a completely determinate radical r in Mod.

Now show that r is an ideally hereditary radical. Let (A, U) be a module of Mod and (B, V) be an ideal of (A, U). Since V is an ideal of the ring U, B is a subgroup of Abelian group A, R is an ideally hereditary radical in *As* and R_0 is a torsion in *Ab*, $R(V) = V \bigcap R(U)$ and $R_0(B) = B \bigcap R_0(A)$. Thus $r(B, V) = (R_0(B), R(V)) = (B \bigcap R_0(A), V \bigcap R(U)) = (B, V) \bigcap r(A, U)$.

The third assertion of the theorem follows from the first two since the expressions given there are mutually inverse.

Definition 7: *The radical r in the sense of Kurosh [1] is called strict if r-radical $r(A, U)$ of any module (A, U) contains all r-submodules of (A, U).*

Definition 8: *The radical r of the category Mod is called* strongly hereditary *if* $r(A_1, U_1) = r(A, U) \bigcap (A_1, U_1)$ *for all submodules (A_1, U_1) of $(A, U) \in Mod$.*

It is easy to prove that every strongly hereditary radical is a strict radical. In the same way we may prove the following theorem.

Theorem 5: *Assume that the strongly hereditary radical r is specified in Mod. Then in As it induces a strongly hereditary radical R and in Ab it induces a torsion R_{Ab} such that the pair $\{R, R_{Ab}\}$ of strongly hereditary radicals is matched. Conversely, every matched pair of strongly hereditary radicals $\{R, R_{Ab}\}$ specifies a completely determinate strongly hereditary radical r in Mod whose radical class includes all modules (A, U) such that $U = R(U)$ and $A = R_{Ab}(A)$. There exists a one-to-one correspondence between all strongly hereditary radicals of Mod and all matched pairs of strongly hereditary radicals.*

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Submitted 23.11.2012, accepted 18.02.2013.

Ռադիկալները և մինչռադիկալները բոլոր օղակների վրա մոդուլների կատեգորիայում

Գ. Էմին-Տերյան

Ամփոփում

Դիտարկվում է բոլոր օղակների վրա մոդուլների Mod կատեգորիան: Այդ կատեգորիայի օբյեկտները բոլոր այնպիսի (A, U) զույգերն են, որտեղ U -ն ասոցիատիվ օղակ է և անպայման չէ, որ ունենա միավոր, իսկ A-ն՝ աջ U -մոդուլ : Ընդսմին, (A, U) մոդուլից դեպի (B, V) մոդուլ մորֆիզմների բազմությունը բաղկացած է (φ_A, φ_U) զույգերից, որտեղ φ_A-ն աբելյան A խմբի հոմոմորֆիզմ է դեպի B խումբ, φ_U-ն U օղակի հոմոմորֆիզմ է դեպի V օղակ և $(a \cdot u)\varphi_A = a\varphi_A \cdot u\varphi_U$, $a \in A$, $u \in U$: Աբելյան խմբերի Ab կատեգորիան և ասոցիատիվ օղակների As կատեգորիան Mod-ի լրիվ ենթակատեգորիաներ են, մինչդեռ $Mod - U$ կատեգորիան ֆիքսված U օղակի վրա աջ մոդուլների կատեգորիան է, որտեղ U -ն ցանկացած ասոցիատիվ օղակ է՝ Mod-ի ենթակատեգորիա: Օգտագործելով Mod կատեգորիայի ենթակատեգորիաների ”ըստ շերտերի” ներկայացման հեղինակի կողմից առաջարկված մեթոդը, հնարավոր դարձավ նկարագրել այդ կատեգորիայի ռադիկալները և մինչռադիկալները: Դրանք նկարագրված են ասոցիատիվ օղակների և ֆիքսված օղակի վրա մոդուլների ռադիկալների, համապատասխանաբար, մինչռադիկալների համաձայնեցված, այսինքն, համապատասխան պայմաններին բավարարող համակարգերի օգնությամբ:

Радикалы и предрадикалы в категории модулей над всеми кольцами

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Аннотация

Рассматривается категория Mod модулей над всеми кольцами. Объекты этой категории - всевозможные пары (A, U), где U - ассоциативное кольцо, не обязательно с единицей, A-правый U-модуль. Множество морфизмов модуля (A, U) в модуль (B, V) состоит из пар (φ_A, φ_U), где φ_A - гомоморфизм абелевой группы A в абелеву группу B, а φ_U - гомоморфизм кольца U в кольцо V, причем $(a \cdot u)\varphi_A = a\ \varphi_A \cdot u\ \varphi_U$, $a \in A$, $u \in U$.

Категория абелевых групп Ab и категория ассоциативных колец As являются полными подкатегориями категории Mod. Для любого ассоциативного кольца U категория $Mod - U$ правых модулей над фиксированным кольцом U является подкатегорией категории Mod.

С помощью предложенного автором метода "послойного" представления исследуемых подкатегорий категории Mod описаны радикалы и предрадикалы этой категории. Они описаны с помощью согласованных, то есть, удовлетворяющих некоторым условиям систем радикалов, соответственно, предрадикалов категорий ассоциативных колец и модулей над фиксированным кольцом.