On Initial Segments of Turing Degrees Containing Simple $T$-Mitotic but not $wtt$-Mitotic Sets

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Abstract

We consider the properties of computably enumerable (c.e.) Turing degrees containing sets, which possess the property of a $T$-mitotic splitting but don’t have a $wtt$-mitotic splitting.

It is proved that for any noncomputable c.e. degree $b$ there exists a degree $a$, such that $a \leq b$ and $a$ contains a simple $T$-mitotic set, which is not $wtt$-mitotic.

Keywords: Mitotic set, $T$-reducibility, $wtt$-reducibility, Simple set, Contiguous degree.

1. Introduction

We shall use the notions and terminology introduced in Soare [1], Rogers [2].

Notations.

We deal with sets and functions over the nonnegative integers $\omega = \{0, 1, 2, \ldots \}$.

Let $\omega_{ev} = \{x : (\exists k)(x = 2k)\}$; $\omega_{od} = \{x : (\exists k)(x = 2k + 1)\}$.

Let $\varphi_e$ be the $e^{th}$ partial computable function in the standard listing (Soare [1, p. 25]).

If $A \subseteq \omega$ and $e \in \omega$, let $\Phi^A_e(x) = \Phi_e(A : x) = \{e\}^A(x)$ (see Soare [1, pp. 48-50]).

$\chi_A$ denotes the characteristic function of $A$, which is often identified with $A$ and written simply as $A(x)$.

Let $\varphi(x) \downarrow$ denotes that $\varphi(x)$ is defined, $\varphi(x) \uparrow$ denotes that $\varphi(x)$ is undefined.

$W_e = \text{dom } \varphi_e = \{x : \varphi_e(x) \downarrow\}$.

$\varphi_{e, at s+1}(x) \downarrow$ denotes $\varphi_{e, s+1}(x) \downarrow \& \varphi_{e, s}(x) \uparrow$.

$x \in W_{e, at s+1}$ denotes $x \in W_{e, s+1} - W_{e, s}$.

In the literature, Turing reducibility is usually taken as the main reducibility. If the word “reducibility” is used without a further explanation, it means, as a rule, Turing reducibility. If the term “degree of unsolvability” is used without a further explanation, the $T$-degree is usually meant.

Definition 1: The use function $u(A; e, x, s)$ is $1+$ (the maximum number used in computation if $\Phi^A_{es}(x) \downarrow$), and $= 0$, otherwise. The use function $u(A; e, x)$ is $u(A; e, x, s)$ if $\Phi^A_{es}(x) \downarrow$ for some $s$, and is undefined if $\Phi^A_{es}(x) \uparrow$.
Definition 2: A is computable in (Turing reducible to) B, written $A \leq_T B$, if $A = \Phi_e^B$ for some $e$ (Soare [1, p. 50]).

Definition 3: A is weak truth-table reducible to B, written $A \leq_{wtt} B$, if $(\exists e) [A = \Phi_e^B \& (\exists \text{computable } f) (f(x) \geq u(B; e, x))]$ (where $u(B; e, x)$ is the use function from Definition 1) (Rogers [2, p. 158]).

Definition 4: If $A$ is a noncomputable computably enumerable (c.e.) set, then a splitting of $A$ is a pair $A_1, A_2$ of disjoint c.e. sets such that $A_1 \cup A_2 = A$ (Downey, Stob [3, p. 4]).

Definition 5: C.e. set $A$ is $T$-mitotic (wtt-mitotic), if there is a splitting $A_1, A_2$ of $A$ such that $A_1 \equiv_T A_2 \equiv_T A (A_1 \equiv_{wtt} A_2 \equiv_{wtt} A)$ (Downey, Stob [3, p. 83]).

Definition 6: (i) A set is immune, if it is infinite but contains no infinite c.e. set. (ii) A set is simple, if $A$ is c.e. and $\overline{A}$ is immune (Soare [1, p. 78]).

Definition 7: A c.e. degree $a$ is contiguous if for every pair $A, B$ of c.e. sets in $a$, $A \equiv_{wtt} B$ (Downey, Stob [3, p. 45]).

Note that each contiguous degree, by definition, doesn’t contain $T$-mitotic sets, which are not $wtt$-mitotic.

Lachlan proved the existence of nonmitotic c.e. set (Lachlan [4]).

Ladner proved the existence of completely mitotic c.e. degree (Ladner [5]).

Ladner and Sasso [6] proved, that for every nonzero c.e. degree $b$ there is a nonzero c.e. degree $a \leq b$ such that $a$ is contiguous (see also Downey, Stob [3]).

Thus, there is an infinite class of contiguous degrees, and these degrees, as we have mentioned, don’t contain $T$-mitotic sets, which are not $wtt$-mitotic.

Ingrassia ([7]) proved the density of nonmitotic c.e. sets (in $R$) (see also Downey, Slaman [8]).

E. J. Griffiths ([9]) proved the following Theorem: There exists a low c.e. degree $u$ such that if $v$ is a c.e. degree and $u \leq v$, then $v$ is not completely mitotic.

2. Preliminaries, Basic Modules

Theorem 1: For any noncomputable c.e. degree $b$ there is a degree $a$ such that $a \leq b$ and $a$ contains a simple $T$-mitotic, but not $wtt$-mitotic set.

Proof. (sketch) Let $h$ be a general computable function that maps $\omega$ to $\omega^2$. Let $(\Psi_i, \psi_i)$ denotes the pair $(\Phi_{i_0}, \varphi_{i_0})$ for all $i$, where $h(i) = (i_0, i_1)$ (note that $\Psi_i$ is $wtt$-reduction with $\psi_i$, denoting the corresponding use function).

It is known (Ladner [10]) that the computably enumerable set $A$ is $T$-mitotic, $\iff A$ is $T$-autoreducible, and similarly, the computably enumerable set $A$ is $wtt$-mitotic, $\iff A$ is $wtt$-autoreducible (Downey, Stob [3], see also Trakhtenbrot [11]).

Therefore, in order to achieve non-$wtt$-mitoticity, it is enough for us to achieve non-$wtt$-autoreducibility.

Thus, to prove our theorem, it is necessary to construct such a c.e. set $A$, so that the following requirements are met.
$R_e : \left( \exists x \right) \neg (\Psi_e(A \cup \{x\}; x) = A(x)), \text{ if } (\forall z \leq y)(\psi_e(z) \downarrow).$

$P_e : (W_e \text{ is infinite}) \Rightarrow (\exists x) \left( x \in W_e \& x \in A \right).$

Note that satisfying the $R_e$ requirement (for all $e$) provides the infinity of the set $A$.

Order the requirements in the following priority ranking: $R_0, P_0, R_1, P_1, \ldots, R_n, P_n, \ldots$

Let $l(e, s) = \max \{ x : (\forall y < x) (\Psi_{e,s}(A_s \cup \{y\}; y) = A(y) \& (\forall z \leq x)(\psi_{e,z} \downarrow)) \}.$

The main strategy for satisfying $R_e$ is the following: we select a number (the so-called follower) $x$ (which should be accompanied by two more elements $x-2$ and $x-1$, and possibly, the third - $\hat{x}$; an exact definition of the attendant numbers of the follower $x$, namely $(x-2), (x-1), \hat{x}$, will be given hereinafter), we wait until $l(e, s) > x$ and enumerate $x$ in $A_{s+1}$, if $(\forall z < x)(\psi_{e,z} \downarrow)$, setting $r(e, s+1) = u(x, e, s)$, where $u(x, e, s) = u(\Psi_{e,s}(A_s \cup \{x\}; x)).$

An $B$-permitting procedure is introduced in order to provide $A \leq_T B$ (where $B$ is a c.e. set from degree $b$).

To satisfy the requirement of $R_e$ at each stage, we have a finite set of followers $x_{1,s}, < \ldots < x_{n,s}$. In this construction, a modification of the $B$-permitting method is used. We treat the interval $[0, \ldots, i]$ as allowing for $x_{i,s}$.

To satisfy the requirement of $P_e$ at each stage, we have a finite set of followers $y_{1,s}, < \ldots < y_{n,s}$. For requirement $P_e$, the usual $B$-permitting method is used.

The construction will be such that if eventually we have $\Psi_e(A \cup \{x\}; x) = A(x) \& \psi_e$ is a total function, then it will be possible to compute $B$.

The ground of satisfactions for requirements of $R_e$ and $P_e$ will be given below.

### 2.1 Basic Module for $R_e$

The followers $x_{1,s}, \ldots, x_{n,s}$ satisfy the following rules below.

**Appointment.** If $x_{i,s}$ is currently defined and $x_{i+1,s}$ is not, then if $l(e, s) > x_{i,s}$, declare $x_{i,s}$ as active, and set $x_{i+1,s} = \mu z (z \geq s + 2 \& (\exists k) (z = 2k))$. Set $\check{r}(e, s+1) = \max(u(x_{k,s}, e, s) : k \leq i)$. To get an idea of the restriction function $\check{r}(e, s)$, we give its definition, although it is not used in the basic module.

We say that $x_{i,s}$ is superactive, if $x_{i,s} - 2$ and $x_{i,s} - 1$ belong to $A_s$.

**Permission.** If $x_{i,s}$ is active and $i \in B_{ats}$, then if

(i) $(\exists j > i) [x_{j,s} \text{ is superactive} \& x_{j,s} \notin A_s]$, let $j_0 = \mu z$ [$x_{z,s} \text{ is superactive} \& x_{z,s} \notin A_z$]. Then we enumerate the numbers $x_{j_0,s}, \hat{x}_{j_0,s}$ into $A_{s+1}$ (where $\hat{x}_{j_0,s} = \psi_e(x_{j_0,s})$). Cancel $x_{k,s}$ for all $(k > j_0)$. We will do the same with the accompanying elements of the corresponding followers.

(ii) if (i) and $(\neg \exists j) [x_{j,s} \in A_s]$ does not hold, then we enumerate the numbers $x_{i,s} - 2, x_{i,s} - 1$ into $A_{s+1}$. Cancel $x_{k,s}$ for all $(k > i)$. We will do the same with the accompanying elements of the corresponding followers.

For any $i$ such that the follower $x_{i,s}$ is not canceled at the end of the part “permission” of the basic module and is active, let’s set $x_{i,s+1} = x_{i,s}$. We will do the same with the accompanying elements of the corresponding followers.

The “cancellation” rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case it will be necessary to note the effect of the requirements of $R_i$ and $P_j$ (where $j < e$) on satisfying the requirement $R_e$, but not to describe the basic module itself for $R_e$. 
2.2 Basic Module for $P_e$

The followers $y_{1,s}, \ldots, y_{n,s}$ satisfy the following rules below.

**Appointment.** If $y_{i,s}$ is currently defined and $y_{i+1,s}$ is not, then if $(\exists z) (z \in W_e \land z \geq y_{i,s})$, declare $y_{i,s}$ as active, and set $y_{i+1,s+1} = \mu z (z \geq s \land (\exists k) (z = 2k))$.

**Permission.** If $y_{i,s}$ is active and $i \in B_{at,s}$, then enumerate the numbers $y_{i,s}, y_{i,s} + 1, z$ and $z + 1$ into $A_{s+1}$.

The “cancellation” rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case it will be necessary to note the effect of the requirements of $R_j$ (where $j < e$) on satisfying the requirement $P_e$, but not to describe the basic module itself for $P_e$.

3. Verification of Lemmas

**Lemma 1:** Suppose that $\psi_e$ is total and $(\forall x) (\Psi_e(A \cup \{x\}; x) \downarrow)$. Then $(\exists y) \neg((\Psi_e(A \cup \{y\}; y) = A(y))$. Thus, the requirement $R_e$ is satisfied.

**Proof.** Suppose otherwise. We show that $B$ is computable.

Note that since we only consider the satisfaction of the basis module for $R_e$ (that is, we do not take into account the effect of the requirements $R_j$ and $P_j$ (where $j < e$) on the satisfaction of the requirement $R_e$), it is obvious that conditions (i), ..., (iv) are met.

(i) All the $x_{i,s}$ eventually become permanently defined, that is $\lim_s x_{i,s} = x_i$ exists with $x_i \notin A$.

(ii) Once $x_k$ is defined at stage $t$, $(\forall s > t) (u(x_k, e, t) = u(x_k, e, s) = u(e, x_k))$.

(iii) $(\forall i) (x_{i+1} > \max \{u(e, x_k) : k \leq i\})$.

(iv) It can be effectively recognized, when (i) occurs.

Two cases are possible:

(a) $(\exists m) (\forall k > m) [x_k - 2 \notin A]$;

(b) $(\forall m) (\exists k > m) [x_k - 2 \in A]$.

For both cases ((a) and (b)), it will be proved that $B$ is computable (and thus, the assumption that Lemma 1 is false will lead to a contradiction with the supposition of non-computability of $B$).

Now, if (a) holds, we prove that $B$ is computable.

If conditions (i), ..., (iv) are satisfied, we show how to compute $B$ (that is, the characteristic function of the set $B$; remind that we often identify the set $B$ with its characteristic function).

Let $f \downarrow x$ denotes the restriction of $f$ to arguments $y < x$, and $A \downarrow x$ denotes $\chi_A \downarrow x$.

Let $s_0$ be such a stage that $B \downarrow m + 1 = B_{s_0} \downarrow m + 1$ and $A \downarrow x_{m+1} = A_{s_0} \downarrow x_{m+1}$.

Let $q \in \omega$ and $q > m$. Effectively compute a stage $s$ so that $x_{q+1}$ is defined, that is $x_{q+1} = x_{q+1,s}$ (in that case, in fact, $s > s_0$).

Then $x_q$ is active, $x_q \in A$ and since $x_{q+1}$ is the final value of the $q + 1$-th follower, the computations of $u(x_j, e, s)$ are true for all $j \leq q$.

In this case $q \in B \iff q \in B_s$, because otherwise it would lead to the fact that $x_q - 2$
would have entered the set, contrary to our assumption that case (a) holds.

Now suppose that case (b) holds. Let us prove that in this case also $B$ is computable.

If conditions (i), ..., (iv) are fulfilled, we show how to compute $B$.

Let $q \in \omega$. Effectively compute such a stage $s$ and a number $p$ so that $p = \mu z \ (z \geq q \ \& \ x_{z-2} \in A \ \& \ x_{z+1} = x_{z+1,s})$.

Then $x_p$ is active, $x_p \notin A$ and since $x_{p+1}$ is the final value of the $p + 1$-th follower, then $u(x_j, e, s)$ computations are true for all $j \leq p$. In this case $q \in B \iff q \in B_e$, since otherwise (that is, if $q$ enters $B$ after the stage $s$) this will lead to the entry $p$ into $A$ and satisfaction of the requirement $R_e$, which will contradict the initial assumption that Lemma 1 is false.

Lemma 1 is prooved.

Lemma 2: Suppose that $W_e$ is an infinite set. Then $(\exists z) \ (z \in W_e \ \& \ z \in A)$. Thus, the requirement $P_e$ is satisfied.

**Proof.** Suppose otherwise.

We show that $B$ is computable.

Let $\hat{r}(e) = \lim_n \hat{r}(e, s)$.

Although the use of this function in the description of the basis module for $P_e$ is not necessary, an indication of this function clearly shows the effect of the requirements $R_j$ (where $j \leq e$) on the satisfaction of the requirements $P_e$ when constructing the set $A$.

Let $t_0$ be such that $(\forall s \geq t_0) \hat{r}(e, s) = \hat{r}(e)$.

Then it is obvious, that all the $y_{i,s}$ become permanently defined (i.e., $\forall i \exists (t \geq t_0) \ (\forall s) (y_{i,t} = y_{i,s} = y_i)$) with $y_i \notin A$.

In fact, if there existed $k$ such that $y_k \in A$, then, by construction, there would exist $z$ such that $z \in W_e \cap A$.

Assuming the opposite of the statement of the proposition, we show how $B$ can be computed.

Let $q \in \omega$. Find $t \geq t_0$ such that $y_q$ is permanently defined. Then $q \in B \iff q \in B_1$, since otherwise $q$’s entry into $B$ would meet $P_e$.

Lemma 2 is prooved.

4. Conclusion

Note that the coherence of constructions to satisfy the requirements $R_e$ and $P_e$ (for all $e$) is not difficult, since satisfying the requirements $R_e$ and $P_e$ (for all $e$) requires a finite number of steps. We also note that the indicated method of constructing the set $A$ (based on the constructions for the basic modules) will result in the set $A \cap \omega_{ev}$ being $T$-equivalent to the set $A \cap \omega_{od}$.

These remarks allow us to complete the proof of the theorem. ■

Note that it follows from the above theorem that below any noncomputable c.e. degree there is an infinite number of noncomputable c.e. degrees with the abovementioned property (since the degree $a$ (mentioned in the theorem), containing a simple set, is a noncomputable c.e. degree).
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References


 Hydra $T$-իրենից, թաղ նե wtt-իրենից բազմաբազմություն պարզապետության հատուկանություն բնորոշման մեջ

Անվան և կենտրոն

Հարցվում է հնարավոր և պարզապետական բազմաբազմություն համար

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Անվան

Հնարավոր է $T$-իրենից բազմաբազմության համար, թաղ wtt-իրենից բազմաբազմության համար $a$ բազմաբազմություն պարզապետության հատուկանություն բնորոշվում է, ուր կոնդակում է $b$ բազմաբազմություն պարզապետության հատուկանություն բնորոշվում է, թաղ $a$ բնորոշվում է, եթե $b \leq a$ և $b$-ը ավելի է, թե դա $T$-իրենից թաղ կոմբինացված


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О некоторых свойствах тьюринговых степеней, содержащих простые $T$-митотические множества, не являющиеся $wtt$-митотическими

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Аннотация

Исследуются свойства рекурсивно перечислимых (р.п.) тьюринговых степеней, содержащих множества, которые обладают свойством -митотического разбиения, но не имеют $wtt$-митотического разбиения. Доказано, что для любой нерекурсивной (р.п.) степени $a$ существует нерекурсивная (р.п.) степень $b$, такая что $b \leq a$ и $b$ содержит простое $T$-митотическое множество, которое не является $wtt$-митотическим.

Ключевые слова: митотическое множество, $T$-сводимость, $wtt$-сводимость, простое множество, сцепленная степень.