Mathematical Problems of Computer Science 41, 23–37, 2014.

On Hamiltonian Bypasses in one Class of Hamiltonian Digraphs

Samvel Kh. Darbinyan and Iskandar A. Karapetyan

Institute for Informatics and Automation Problems of NAS RA
e-mail: samdarbin@ipia.sci.am, isko@ipia.sci.am

Abstract

Let D be a strongly connected directed graph of order $n \geq 4$ which satisfies the following condition (*): for every pair of non-adjacent vertices x, y with a common in-neighbour $d(x) + d(y) \geq 2n - 1$ and $min\{d(x), d(y)\} \geq n - 1$. In [2] (J. of Graph Theory 22 (2) (1996) 181-187)) J. Bang-Jensen, G. Gutin and H. Li proved that D is Hamiltonian. In [9] it was shown that if D satisfies the condition (*) and the minimum semi-degree of D at least two, then either D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n - 1$) or n is even and D is isomorphic to complete bipartite digraph (or to complete bipartite digraph minus one arc) with equal partite sets. In this paper we show that if the minimum out-degree of D at least two and the minimum in-degree of D at least three, then D contains also a Hamiltonian bypass, (i.e., a subdigraph is obtained from a Hamiltonian cycle by reversing exactly one arc).

Keywords: Digraphs, Cycles, Hamiltonian cycles, Hamiltonian bypasses.

1. Introduction

The directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle that includes every vertex of D. A Hamiltonian bypass in D is a subdigraph obtained from a Hamiltonian cycle by reversing exactly one arc. We recall the following well-known degree conditions (Theorems 1-5) that guarantee that a digraph is Hamiltonian.

Theorem 1: (Nash-Williams [14]). *Let D be a digraph of order n such that for every vertex x, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.*

Theorem 2: (Ghouila-Houri [12]). *Let D be a strong digraph of order n. If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 3: (Woodall [16]). *Let D be a digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 4: (Meyniel [13]). *Let D be a strong digraph of order $n \geq 2$. If $d(x)+d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

It is easy to see that Meyniel's theorem is a common generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.3, see [5].

C. Thomassen [15] (for $n = 2k+1$) and S. Darbinyan [6] (for $n = 2k$) proved the following:

Theorem 5: [15, 6]. *If D is a digraph of order $n \geq 5$ with minimum degree at least $n-1$ and* *with minimum semi-degree at least* $n/2-1$*, then* D *is Hamiltonian (unless some extremal cases which are characterized).*

In view of the next theorems we need the following definitions.

Definition 1: *Let* D_0 *denote any digraph of order* $n \geq 5$*,* n *odd, such that* $V(D_0) = A \cup B$*, where* $A \cap B = \emptyset$*,* A *is an independent set with* $(n+1)/2$ *vertices,* B *is a set of* $(n-1)/2$ *vertices inducing any arbitrary subdigraph, and* D_0 *has* $(n+1)(n-1)/2$ *arcs between* A *and* B*. Note that* D_0 *has no Hamiltonian bypass.*

Definition 2: *For any* $k \in [1, n-2]$ *let* D_1 *denote a digraph of order* $n \geq 4$*, obtained from* K^*_{n-k} *and* K^*_{k+1} *by identifying a vertex of the first with a vertex of the second. Note that* D_1 *has no Hamiltonian bypass.*

Definition 3: *By* $T(5)$ *we denote a tournament of order 5 with vertex set* $V(T(5)) = \{x_1, x_2, x_3, x_4, y\}$ *and arc set* $A(T(5)) = \{x_i x_{i+1}/i \in [1,3]\} \cup \{x_4x_1, x_1y, x_3y, yx_2, yx_4, x_1x_3, x_2x_4\}$*.* $T(5)$ *has no Hamiltonian bypass.*

In [4] it was proved that if a digraph D satisfies the condition of Nash-Williams' or Ghouila-Houri's or Woodall's theorem, then D contains a Hamiltonian bypass. In [4] the following theorem was also proved:

Theorem 6: (Benhocine [4]).*Every strongly 2-connected digraph of order* n *and with minimum degree at least* $n-1$ *contains a Hamiltonian bypass, unless* D *is isomorphic to a digraph of type* D_0*.*

In [7] the first author proved the following theorem:

Theorem 7: (Darbinyan [7]).*Let* D *be a strong digraph of order* $n \geq 3$*. If* $d(x)+d(y) \geq 2n-2$ *for all pairs of non-adjacent vertices in* D*, then* D *contains a Hamiltonian bypass unless it is isomorphic to a digraph of the set* $D_0 \cup \{D_1, T_5, C_3\}$*, where* C_3 *is a directed cycle of length 3.*

For $n \geq 3$ and $k \in [2, n]$, $D(n, k)$ denotes the digraph of order n obtained from a directed cycle C of length n by reversing exactly $k-1$ consecutive arcs. The first author [7, 8] has studied the problem of the existence of $D(n, 3)$ in digraphs with the condition of Meyniel's theorem and in oriented graphs with large in-degrees and out-degrees.

Theorem 8: (Darbinyan [7]). *Let* D *be a strong digraph of order* $n \geq 4$*. If* $d(x) + d(y) \geq 2n-1$ *for all pairs of non-adjacent vertices in* D*, then* D *contains a* $D(n, 3)$*.*

Theorem 9: (Darbinyan [8]). *Let* D *be an oriented graph of order* $n \geq 10$*. If the minimum in-degree and out-degree of* D *at least* $(n-3)/2$*, then* D *contains a* $D(n, 3)$*.*

Each of Theorems 1-5 imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). The following theorem (as well as Theorems 13 and 14) imposes a degree condition only for some pairs of non-adjacent vertices.

Theorem 10: [2] *(Bang-Jensen, Gutin, H.Li [2]). Let* D *be a strong digraph of order* $n \geq 2$*. Suppose that*

$$min\{d(x), d(y)\} \geq n-1 \quad and \quad d(x)+d(y) \geq 2n-1 \qquad (*)$$

for every pair of non-adjacent vertices x, y *with a common in-neighbour, then* D *is Hamiltonian.*

In [9] the following results were obtained:

Theorem 11: [9]. *Let* D *be a strong digraph of order* $n \geq 3$ *with the minimum semi-degree of* D *at least two. Suppose that* D *satisfies the condition (*). Then either* D *contains a pre-Hamiltonian cycle or* n *is even and* D *is isomorphic to the complete bipartite digraph or to the complete bipartite digraph minus one arc with partite sets of cardinalities* $n/2$ *and* $n/2$*.*

In this paper using Theorem 11 we prove the following:

Theorem 12: (Main Result). Let D be a strong digraph of order $n \geq 4$ with the minimum out-degree at least two and with minimum in-degree at least three. Suppose that

$$min\{d(x), d(y)\} \geq n-1 \quad \text{and} \quad d(x)+d(y) \geq 2n-1 \tag{$*$}$$

for every pair of non-adjacent vertices x, y with a common in-neighbour. Then D contains a Hamiltonian bypass.

2. Terminology and Notations

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraphs) and refer the reader to [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The order of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy or $x \rightarrow y$. If x, y, z are distinct vertices in D, then $x \rightarrow y \rightarrow z$ denotes that xy and $yz \in D$. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). By $a(x, y)$ we denote the number of arcs with end vertices x and y, in particular, $a(x, y)$ means that the vertices x and y are non-adjacent. For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two distinct subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \rightarrow B$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighborhood of x. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A/xy \in A(D)\}$ and $N^-(x, A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, by $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. For a cycle $C_k := x_1 x_2 \cdots x_k x_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. For an undirected graph G, we denote by G^* the symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinalities p and q. For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b. By $D(n; 2) = [x_1 x_n; x_1 x_2 \ldots x_n]$ is denoted the Hamiltonian bypass obtained from a Hamiltonian cycle $x_1 x_2 \ldots x_n x_1$ by reversing the arc $x_n x_1$.

3. Preliminaries

The following well-known simple Lemmas 1 and 2 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proof of our result.

Lemma 1: [11].*Let D be a digraph of order $n \geq 3$ containing a cycle C_m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m+1$, then D contains a cycle C_k for all $k \in [2, m+1]$.*

The following lemma is a slight modification of a lemma by Bondy and Thomassen [5].

Lemma 2: *Let D be a digraph of order $n \geq 3$ containing a path $P := x_1x_2 \ldots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:*

(i) $d(x, P) \geq m+2$;

(ii) $d(x, P) \geq m+1$ and $xx_1 \notin D$ or $x_mx_1 \notin D$;

(iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_mx \notin D$,

then there is an $i \in [1, m-1]$ such that $x_ix, xx_{i+1} \in D$ (the arc x_ix_{i+1} is a partner of x), i.e., D contains a path $x_1x_2 \ldots x_ixx_{i+1} \ldots x_m$ of length m (we say that x can be inserted into P or the path $x_1x_2 \ldots x_ixx_{i+1} \ldots x_m$ is extended from P with x).

Definition 4: ([1], [2]). *Let $Q = y_1y_2 \ldots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \ldots x_t$, $t \geq 2$, be a path in $D - V(Q)$. Q has a partner on P if there is an arc (the partner of Q) x_ix_{i+1} such that $x_iy_1, y_sx_{i+1} \in D$. In this case the path Q can be inserted into P to give a new (x_1, x_t)-path with vertex set $V(P) \cup V(Q)$. The path Q has a collection of partners on P if there are integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ such that, for every $k = 2, 3, \ldots, m$ the subpath $Q[y_{i_{k-1}}, y_{i_k - 1}]$ has a partner on P.*

Lemma 3: ([1], [2], Multi-Insertion Lemma). *Let $Q = y_1y_2 \ldots y_s$ be a path in a digraph D (possibly, $s = 1$) and let $P = x_1x_2 \ldots x_t$, $t \geq 2$, be a path in $D - V(Q)$. If Q has a collection of partners on P, then there is an (x_1, x_t)-path with vertex set $V(P) \cup V(Q)$.*

The following lemma is obvious.

Lemma 4: *Let D be a digraph of order $n \geq 3$ and let $C := x_1x_2 \ldots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. If D contains no Hamiltonian bypass, then*

(i) $d^+(y, \{x_i, x_{i+1}\}) \leq 1$ and $d^-(y, \{x_i, x_{i+1}\}) \leq 1$ for all $i \in [1, n-1]$;

(ii) $d^+(y) \leq (n-1)/2$, $d^-(y) \leq (n-1)/2$ and $d(y) \leq n-1$;

(iii) if $x_ky, yx_{k+1} \in D$, then $x_{i+1}x_i \notin D$ for all $x_i \neq x_k$.

Let D be a digraph of order $n \geq 3$ and let C_{n-1} be a cycle of length $n-1$ in D. If for the vertex $y \notin C_{n-1}$, $d(y) \geq n$, then we say that C_{n-1} is a good cycle. Notice that, by Lemma 4(ii), if a digraph D contains a good cycle, then D also contains a Hamiltonian bypass.

We now need to state and prove some general lemmas.

Lemma 5: *Let D be a digraph of order $n \geq 6$ with minimum semi-degree at least two satisfying the condition (*). Let $C := x_1x_2 \ldots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. Then for any $i \in [1, n-1]$ the following holds:*

(i) If $yx_i \notin D$ and $x_{i-2}x_i \notin D$, then x_i has a partner on $C[x_{i+1}, x_{i-2}]$ or $d(x_i) \geq n-1$.

(ii) If $yx_i \notin D$ and $d(x_i) \leq n-2$, then x_i has a partner on $C[x_{i+1}, x_{i-2}]$ or there is a vertex $x_k \in C[x_{i+1}, x_{i-2}]$ such that $\{x_k, x_{k+1}, \ldots, x_{i-2}\} \rightarrow x_i$.

(iii) If $yx_i \in D$, $x_{i-2}x_i \notin D$ and $d^-(x_i) \geq 3$, then x_i has a partner on $C[x_{i+1}, x_{i-1}]$ or $d(x_i) \geq n-1$.

Proof: (i) The proof is by contradiction. Assume that x_i has no partner on $C[x_{i+1}, x_{i-2}]$ and $d(x_i) \leq n-2$. Since $d^-(x_i, \{y, x_{i-2}\}) = 0$ and $d^-(x_i) \geq 2$, there is an $x_k \in C[x_{i+1}, x_{i-3}]$ such that $x_kx_i \in D$. From $x_k \rightarrow \{x_{k+1}, x_i\}$, $d(x_i) \leq n-2$, $x_ix_{k+1} \notin D$ and the condition (*) it follows that $x_{k+1}x_i \in D$. By a similar argument we conclude that $x_{i-2}x_i \in D$, which is a contradiction.

For the proofs of (ii) and (iii) we can use precisely the same arguments as in the proof of (i). □

Lemma 6: *Let D be a digraph of order $n \geq 6$ with minimum semi-degree at least two satisfying the condition (*). Let $C := x_1x_2 \ldots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. Then*

(i) If for some $i \in [1, n-1]$, $x_iy \in D$ and x_{i+1}, y are non-adjacent or (ii) $a(x_i, y) = 2$ or

(iii) $d(y) \geq n-1$, then D contains a Hamiltonian bypass.

Proof: (i) Assume that (i) is not true. Without loss of generality, we assume that $x_{n-1}y \in D$, $d(y, \{x_1, x_2, \ldots, x_a\}) = 0$ and x_{a+1}, y are non-adjacent, where $a \geq 1$. Then x_1 and y is a dominated pair of non-adjacent vertices with a common in-neighbour x_{n-1}. Therefore, by condition (*), $d(y) \geq n-1$ and $d(x_1) \geq n-1$. On the other hand, using Lemma 4(i) we obtain that $d(y) \leq n-a$ and hence, $a = 1$ and $d(y) = n-1$. This together with condition (*) implies that $d(x_1) \geq n$. If $yx_2 \in D$, then $x_{n-1}yx_2x_3 \ldots x_{n-2}x_{n-1}$ is a good cycle in D and therefore, D contains a Hamiltonian bypass. Assume therefore that $x_2y \in D$. Since $d(x_1) \geq n$, by Lemma 2, x_1 has a partner on the path $C[x_2, x_{n-1}]$, i.e, there is an (x_2, y)-Hamiltonian path which together with the arc x_2y forms a Hamiltonian bypass, which is a contradiction and completes the proof of (i).

(ii) It follows immediately from Lemmas 6(i) and 4(i).

(iii) Suppose, on the contrary, that $d(y) \geq n-1$ and D contains no Hamiltonian bypass as well as no good cycle. By Lemma 4(ii), $d(y) = n-1$. From Lemma 6(ii) it follows that $a(y, x_i) = 1$ for all $i \in [1, n-1]$. Without loss of generality, we may assume that (by Lemma 4(i))

$$N^+(y) = \{x_1, x_3, \ldots, x_{n-2}\} \quad \text{and} \quad N^-(y) = \{x_2, x_4, \ldots, x_{n-1}\}. \tag{1}$$

Notice that

(2) for every vertex x_i, $x_ix_{i-1} \notin D$ and x_i has no partner on the path $C[x_{i+1}, x_{i-1}]$ (for otherwise, D contains a Hamiltonian bypass).

Assume first that $x_1x_3 \in D$. Then it is not difficult to show that x_2, x_{n-1} are non-adjacent and $x_2x_4 \notin D$. Indeed, by (1) if $x_{n-1}x_2 \in D$, then $D(n,2) = [x_1x_2; x_1x_3x_4yx_5 \ldots x_{n-1}x_2]$; if $x_2x_4 \in D$, then $D(n,2) = [x_2x_3; x_2x_4x_5 \ldots x_{n-1}yx_1x_3]$; and if $x_2x_{n-1} \in D$, then $D(n,2) = [x_2x_{n-1}; x_2yx_1x_3x_4 \ldots x_{n-1}]$, in each case we have a contradiction. Now, since $x_{n-1}x_2 \notin D$, $yx_2 \notin D$ and x_2 has no partner on $C[x_3, x_1]$, Lemma 5(i) implies that $d(x_2) \geq n-1$. On the other hand, using Lemma 2(ii), $a(x_2, x_{n-1}) = 0$, $x_2x_4 \notin D$ and (2), we obtain

$$n-1 \leq d(x_2) = d(x_2, \{y, x_1, x_3\}) + d(x_2, C[x_4, x_{n-2}]) \leq n-2,$$

a contradiction.

Assume second that $x_1x_3 \notin D$. By the symmetry of the vertices x_{n-2} and x_1 (by (1)), we also may assume that $x_{n-2}x_1 \notin D$. Since x_1 has no partner on $C[x_3, x_{n-2}]$, again using Lemma 2(iii) and (2) we obtain

$$d(x_1) = d(x_1, \{x_{n-1}, x_2, y\}) + d(x_1, C[x_3, x_{n-2}]) \leq n-2.$$

Therefore, by condition (*), we have that x_1 is adjacent with x_3 and x_{n-2}, i.e., $x_3x_1, x_1x_{n-2} \in D$, since $y \to \{x_{n-2}, x_1, x_3\}$. Now it is easy to see that $\{x_3, x_4, \ldots, x_{n-2}\} \to x_1$, which contradicts that $x_{n-2}x_1 \notin D$. In each case we obtain a contradiction, and hence, the proof of Lemma 6(iii) is completed. □

The following simple observation is of importance in the rest of the paper.

Remark: *Let D be a digraph of order $n \geq 6$ with minimum semi-degree at least two satisfying* *the condition (*). Let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. If D contains no Hamiltonian bypass, then*

(i) There are two distinct vertices x_k and x_l such that $\{x_k, x_{k+1}\} \cap \{x_l, x_{l+1}\} = \emptyset$, $x_k \to y \to x_{k+1}$ and $x_l \to y \to x_{l+1}$ (by Lemmas 4(i) and 5(i)).

(ii) $x_{i+1}x_i \notin D$ for all $i \in [1, n-1]$.

(iii) If $y \to \{x_{i-1}, x_{i+1}\}$ or $\{x_{i-1}, x_{i+1}\} \to y$, then x_i has no partner on the path $C[x_{i+1}, x_{i-1}]$.

(iv) If $x_{i+1}x_{i-1} \in D$, then $d(x_i) \leq n-2$ (by Remark (i) and Lemma 6(ii)).

Lemma 7: *Let D be a digraph of order $n \geq 6$ with minimum semi-degree at least two satisfying the condition (*). Let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. Assume that $y \to \{x_2, x_{n-1}\}$, $x_1 \to y$ and $d(y, \{x_3, x_{n-2}\}) = 0$. Then D contains a Hamiltonian bypass.*

Proof: The proof is by contradiction. Assume that D contains no Hamiltonian bypass. From Remark (i) and Lemmas 6(i), 4(i) it follows that for some $j \in [4, n-4]$, $x_j \to y \to x_{j+1}$.

Now we show that $x_{n-2}x_1 \notin D$. Assume that this is not the case. Then $x_{n-2}x_1 \in D$ and $d(x_{n-1}) \leq n-2$ (by Remark (iv)). Then, since $y \to \{x_2, x_{n-1}\}$, the condition (*) implies that x_2 and x_{n-1} are adjacent, i.e., $x_2x_{n-1} \in D$ or $x_{n-1}x_2 \in D$. If $x_2x_{n-1} \in D$, then $D(n,2) = [x_2x_{n-1}; x_2x_3 \dots x_{n-2}x_1yx_{n-1}]$, and if $x_{n-1}x_2 \in D$, then $D(n,2) = [x_{n-1}x_1; x_{n-1}x_2x_3 \dots x_jyx_{j+1} \dots x_{n-2}x_1]$. In both cases we have a Hamiltonian bypass, a contradiction. Therefore $x_{n-2}x_1 \notin D$.

Now, since x_1 has no partner on $C[x_3, x_{n-2}]$, by Lemma 5(i), $d(x_1) \geq n-1$. On the other hand, from $d(y, \{x_3, x_{n-2}\}) = 0$, $d(y) \leq n-2$ and the condition (*) it follows that $x_1x_3 \notin D$ and $x_1x_{n-2} \notin D$ (in particular, $a(x_1, x_{n-2}) = 0$). Now using Lemma 2(ii) and Remark (ii) we obtain

$$n-1 \leq d(x_1) = d(x_1, \{y, x_2, x_{n-1}\}) + d(x_1, C[x_3, x_{n-3}]) \leq n-2,$$

which is a contradiction. Lemma 7 is proved.

Lemma 8: *Let D be a digraph of order $n \geq 6$ with minimum semi-degree at least two satisfying the condition (*). Let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. If $d^-(y) \geq 3$ and y is adjacent with four consecutive vertices of the cycle C, then D contains a Hamiltonian bypass.*

Proof: Suppose, on the contrary, that D contains no Hamiltonian bypass and no good cycle. Using Lemmas 6(i) and 4(i), without loss of generality, we can assume that $\{x_{n-1}, x_2\} \to y$ and $y \to \{x_1, x_3\}$. By Remarks (ii) and (iii) we have

(3) $x_ix_{i-1} \notin D$ for each $i \in [1, n-1]$ and x_1 (respectively, x_2) has no partner on the path $C[x_2, x_{n-1}]$ (respectively, $C[x_3, x_1]$).

If $x_{n-2}x_1 \notin D$ and $x_1x_3 \notin D$, then using Lemma 2(iii) and (3) we obtain that

$$d(x_1) = d(x_1, \{y, x_2, x_{n-1}\}) + d(x_1, C[x_3, x_{n-2}]) \leq n-2.$$

Therefore, by condition (*), the vertices x_1, x_3 are adjacent, since $y \to \{x_1, x_3\}$ (x_1 and x_3 has a common in-neighbour y). This means that $x_3x_1 \in D$. Since x_1 has no partner on $C[x_3, x_{n-2}]$, it follows from $x_3 \to \{x_1, x_4\}$ and $d(x_1) \leq n-2$ that $x_4x_1 \in D$. Similarly, we conclude that $x_{n-2}x_1 \in D$ which contradicts the assumption that $x_{n-2}x_1 \notin D$. Assume therefore that

$$x_{n-2}x_1 \in D \quad \text{or} \quad x_1x_3 \in D. \tag{4}$$

Now we prove that $d(x_1) \geq n-1$. Assume that this is not the case, that is $d(x_1) \leq n-2$. Then again by condition (*) x_1, x_3 are adjacent because of $y \to \{x_1, x_3\}$.

Therefore $x_3x_1 \in D$ or $x_1x_3 \in D$. If $x_3x_1 \in D$, then it is not difficult to show that $\{x_3, x_4, \dots, x_{n-2}\} \to x_1$, i.e., $d(x_1) \geq n-1$, a contradiction. Assume therefore that $x_3x_1 \notin D$ and $x_1x_3 \in D$. Then $C' := x_1x_3x_4 \dots x_{n-1}yx_1$ is a cycle of length $n-1$ missing the vertex x_2. Then $d(x_2) \leq n-2$ (by Remark (iv)). Now, since x_2 has no partner on $C[x_3, x_1]$ (by (3)) and $d^-(x_2, \{x_i, x_{i+1}\}) \leq 1$ for all $i \in [3, n-2]$ (by Lemma 4(i)), it follows that $d^-(x_2, \{y, x_3, x_4, \dots, x_{n-2}\}) = 0$. Then $x_{n-1}x_2 \in D$ because of $d^-(x_2) \geq 2$. From $d^-(y) \geq 3$ and Lemma 6(i) it follows that there is a vertex $x_j \in C[x_4, x_{n-3}]$ such that $x_j \to y \to x_{j+1}$. Therefore $D(n, 2) = [x_1x_2; x_1x_3x_4 \dots x_jyx_{j+1} \dots x_{n-1}x_2]$ is a Hamiltonian bypass, a contradiction. This contradiction proves that $d(x_1) \geq n-1$.

Notice that $x_{n-1}x_2 \notin D$, by Remark (iv). From (4) it follows that the following two cases are possible: $x_1x_3 \in D$ (Case 1) or $x_1x_3 \notin D$ and $x_{n-2}x_1 \in D$ (Case 2).

Case 1. $x_1x_3 \in D$. Then $d(x_2) \leq n-2$ (by Remark (iv)). It is easy to see that $x_2x_4 \notin D$ and $x_{n-1}x_2 \notin D$ (if $x_{n-1}x_2 \in D$, then D has a cycle of length $n-1$ missing x_1, and hence $d(x_1) \leq n-2$ which contradicts that $d(x_1) \geq n-1$). Thus, we have a contradiction against Lemma 5(i), since $d(x_2) \leq n-2$, $x_{n-1}x_2 \notin D$ and x_2 has no partner on $C[x_3, x_{n-1}]$ (by (3)).

Case 2. $x_1x_3 \notin D$ and $x_{n-2}x_1 \in D$. It is easy to see that $x_{n-1}x_2 \notin D$ and $x_{n-3}x_{n-1} \notin D$. If $yx_{n-2} \in D$, then x_{n-1} has no partner on $C[x_1, x_{n-2}]$. This together with $d(x_{n-1}) \leq n-2$, and $x_{n-3}x_{n-1} \notin D$ contradicts Lemma 5(i). Assume therefore that y and x_{n-2} are non-adjacent. Then, since $d(y) \leq n-2$ and $x_2y \in D$, we have that $x_2x_{n-2} \notin D$.

Assume first that $x_2x_{n-1} \in D$. Then $x_{n-2}x_2 \notin D$ (for otherwise the arc $x_{n-2}x_{n-1} \in C[x_3, x_{n-1}]$ is a partner of x_2 on $C[x_3, x_{n-1}]$, a contradiction against (3)). Therefore x_2 and x_{n-2} are non-adjacent. Now we have $x_ix_2 \in D$, where $i \in [4, n-3]$ since $d^-(x_2) \geq 2$ and $d^-(x_2, \{y, x_3, x_{n-2}, x_{n-1}\}) = 0$. It is not difficult to see that $d(x_2) \geq n-1$ (Lemma 5(i)). Then by Remark (ii) and Lemma 2 we obtain

$$n-1 \leq d(x_2) = d(x_2, \{y, x_1, x_3, x_{n-1}\}) + d(x_2, C[x_4, x_{n-3}]) \leq n-1,$$

i.e., $d(x_2) = n-1$ and $d(x_2, C[x_4, x_{n-3}]) = n-5$. By Lemma 2, x_2x_4 and $x_{n-3}x_2 \in D$. From $d(x_2) = n-1$ and the condition (*) it follows that $d(x_{n-2}) \geq n$, since x_2 and x_{n-2} are non-adjacent and have a common in-neighbour x_{n-3}. If $x_1x_{n-2} \in D$, then $D(n, 2) = [x_1x_2; x_1x_{n-2}x_{n-1}yx_3x_4 \dots x_{n-3}x_2]$, a contradiction. Assume therefore that $x_1x_{n-2} \notin D$. Now we consider the cycle $C' := x_{n-3}x_2x_{n-1}yx_3x_4 \dots x_{n-3}$ of length $n-2$ which does not contain the vertices x_{n-2} and x_1. Since $d(x_{n-2}) \geq n$ and $x_1x_{n-2} \notin D$ (i.e., $a(x_1, x_{n-2}) = 1$), then $d(x_{n-2}, C') \geq n-1$. Therefore, by Lemma 1, there is a cycle, say C'', of length $n-1$ missing the vertex x_1. Then, since $d(x_1, C'') \geq n-1$, by Lemma 4(ii) D contains a Hamiltonian bypass.

Assume second that x_2 and x_{n-1} are non-adjacent. Then, since $d(x_{n-1}) \leq n-2$, the condition (*) implies that $x_{n-2}x_2 \notin D$. Then by Remark (ii) and Lemma 2(ii), we have

$$d(x_2) = d(x_2, \{y, x_1, x_3\}) + d(x_2, C[x_4, x_{n-2}]) \leq n-2.$$

This contradicts Lemma 5(i) (because of (3)) and completes the proof of Lemma 8. □

From Lemmas 6, 7 and 8 immediately the following lemma follows:

Lemma 9: *Let D be a digraph of order $n \geq 6$ with minimum out-degree at least two and with minimum in-degree at least three satisfying the condition (*). Let $C := x_1x_2 \dots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. If the vertex y is adjacent with three consecutive vertices of the cycle C, then D contains a Hamiltonian bypass.*

Lemma 10: *Let D be a digraph of order $n \geq 6$ with minimum out-degree at least two and with minimum in-degree at least three satisfying the condition (*). Let $C := x_1 x_2 \dots x_{n-1} x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. If D contains no Hamiltonian bypass and $x_{i-1}x_{i+1} \in D$ for some $i \in [1, n-1]$, then $d(x_i, \{x_{i-2}, x_{i+2}\}) = 0$.*

Proof: The proof is by contradiction. Without loss of generality, we may assume that D has no Hamiltonian bypass, $x_{n-1}x_2 \in D$ and $a(x_1, x_3) \geq 1$ or $a(x_1, x_{n-2}) \geq 1$. If $a(x_1, x_3) \geq 1$ (respectively, $a(x_1, x_{n-2}) \geq 1$), then, since y is not adjacent with three consecutive vertices of C, by Remark (i) there exists a vertex $x_k \in C[x_3, x_{n-2}]$ (respectively, $x_k \in C[x_2, x_{n-3}]$) such that $x_k \to y \to x_{k+1}$. It is not difficult to see that $C' := C[x_2, x_k]yC[x_{k+1}, x_{n-1}]x_2$ is a cycle of length $n-1$ missing the vertex x_1, and x_1 is adjacent with three consecutive vertices of C', namely with x_{n-1}, x_2, x_3 (respectively, x_{n-2}, x_{n-1}, x_2), which is a contradiction against Lemma 9. Lemma 10 is proved.

Lemma 11: *Let D be a digraph of order $n \geq 6$ with minimum out-degree at least two and with minimum in-degree at least three satisfying the condition (*). Let $C := x_1 x_2 \dots x_{n-1} x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. D contains no Hamiltonian bypass, then $x_{i+1}x_{i-1} \notin D$ for all $i \in [1, n-1]$.*

Proof: The proof is by contradiction. Without loss of generality, we may assume that $x_3 x_1 \in D$.

Assume first that the vertex x_2 has a partner on $C[x_4, x_{n-1}]$, i.e., there is an $x_j \in C[x_4, x_{n-2}]$ such that $x_j \to x_2 \to x_{j+1}$. From $d^-(y) \geq 3$ and Lemma 6(i) it follows that there exists a vertex $x_k \in C[x_3, x_{n-2}]$ distinct from x_j such that $x_k \to y \to x_{k+1}$. Therefore, if $k \geq j+1$, then $D(n, 2) = [x_3 x_1; x_3 x_4 \dots x_j x_2 x_{j+1} \dots x_k y x_{k+1} \dots x_{n-1} x_1]$, and if $k \leq j-1$, then $D(n, 2) = [x_3 x_1; x_3 x_4 \dots x_k y x_{k+1} \dots x_j x_2 x_{j+1} \dots x_{n-1} x_1]$, a contradiction.

Assume second that x_2 has no partner on $C[x_4, x_{n-1}]$. Since $x_3 x_1 \in D$, Lemma 10 implies that $x_2 x_4 \notin D$ and $x_{n-1} x_2 \notin D$. Now using Lemma 2(iii) and Remark (ii) we obtain

$$d(x_2) = d(x_2, \{y, x_1, x_3\}) + d(x_2, C[x_4, x_{n-1}]) \leq n-2.$$

This together with the condition (*) implies that $d^-(x_2, C[x_3, x_{n-1}]) = 0$. Therefore $d^-(x_2) \leq 2$, which contradicts that $d^-(x_2) \geq 3$. Lemma 11 is proved. □

4. The Proof of the Main Result

Proof of Theorem 12. By Theorem 11 the digraph D contains a cycle of length $n-1$ or n is even and D is isomorphic to the complete bipartite digraph (or to the complete bipartite digraph minus on arc) with equal partite sets. If $n \leq 5$ or D contains no cycle of length $n-1$, then it is not difficult to check that D contains a Hamiltonian bypass. Assume therefore that $n \geq 6$, D contains a cycle of length $n-1$ and has no Hamiltonian bypass. From Lemma 9 it follows that if C is an arbitrary cycle of length $n-1$ in D and the vertex y is not on C, then there are not three consecutive vertices of C which are adjacent with y. Let $C := x_1 x_2 \dots x_{n-1} x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C. Then, by Lemma 6(i), the following two cases are possible: There is a vertex x_i and an integer $a \geq 1$ such that $d(y, \{x_{i+1}, x_{i+2}, \dots, x_{i+a}\}) = 0$, $x_{i-1} \to y \to x_i$ and the vertices y, x_{i+a+1} are adjacent (Case I) or $d(y, \{x_{i+1}, x_{i+2}, \dots, x_{i+a}\}) = 0$, $y \to \{x_i, x_{i+a+2}\}$, $x_{i+a+1} y \in D$ and the vertices y, x_{i-1} are non-adjacent, where $a \in [1, n-6]$.

The proof will be by induction on a. We will first show that the theorem is true for $a = 1$.

Case I. $a = 1$. Without loss of generality, we may assume that $x_{n-2} \to y \to x_{n-1}$, x_2, y are adjacent and y , x_1 are non-adjacent. Since the vertex y is not adjacent with three consecutive vertices of C (Lemma 9), it follows that y, x_{n-3} also are non-adjacent. Condition (*) implies that $x_{n-2}x_1 \notin D$, since $d(y) \leq n-2$ and $x_{n-2}y \in D$.

We show that x_1 has a partner on $C[x_3, x_{n-2}]$. Assume that this is not the case. Then by Lemma 5(i) we have $d(x_1) \geq n-1$, since $x_{n-2}x_1 \notin D$ and $yx_1 \notin D$. On the other hand, using Lemma 2(ii) and Remark (ii), we obtain

$$n - 1 \leq d(x_1) = d(x_1, \{x_2, x_{n-1}\}) + d(x_1, C[x_3, x_{n-2}]) \leq n - 2,$$

which is a contradiction.

So, indeed x_1 has a partner on $C[x_3, x_{n-2}]$. Let the arc $x_k x_{k+1} \in C[x_3, x_{n-2}]$ be a partner of x_1, i.e., $x_k \to x_1 \to x_{k+1}$. Notice that $k \in [4, n-4]$ (by Lemma 11). If $yx_2 \in D$, then $D(n, 2) = [yx_{n-1}; yx_2x_3 \dots x_kx_1x_{k+1} \dots x_{n-2}x_{n-1}]$, a contradiction. Assume therefore that $yx_2 \notin D$. Then x_2y, $yx_3 \in D$ and y, x_4 are non-adjacent, by Lemmas 9 and 6(i). This together with the condition (*) implies that $x_2x_4 \notin D$, since $x_2y \in D$ and $d(y) \leq n-2$. If $x_{n-2}x_2 \in D$, then $C' := x_{n-2}x_2yx_3 \dots x_kx_1x_{k+1} \dots x_{n-2}$ is a cycle of length $n-1$ missing the vertex x_{n-1} for which $\{x_{n-2}, y\} \to x_{n-1}$. Then $x_{n-1}x_2 \in D$, by Lemmas 6(i) and 4, i.e., x_{n-1} is adjacent with three consecutive vertices of C', which is contrary to Lemma 9. Assume therefore that $x_{n-2}x_2 \notin D$. Now we show that x_2 also has a partner on $C[x_3, x_{n-2}]$. Assume that this is not the case. Then, since $x_2x_{n-1} \notin D$ (by Lemma 11) and $x_2x_4 \notin D$, using Lemma 2(iii) and Remark (ii) we obtain

$$d(x_2) = d(x_2, \{y, x_1, x_3, x_{n-1}\}) + d(x_2, C[x_4, x_{n-2}]) \leq n - 2.$$

This together with $x_1 \to \{x_2, x_{k+1}\}$ and the condition (*) implies that x_2, x_{k+1} are adjacent. It is easy to see that $x_{k+1}x_2 \in D$. By a similar argument, we conclude that $x_{n-2}x_2 \in D$, which contradicts the fact that $x_{n-2}x_2 \notin D$. Thus, x_2 also has a partner on $C[x_3, x_{n-2}]$. Therefore, by Multi-Insertion Lemma there is a (x_3, x_{n-1})-path with vertex set $V(C)$, which together with the arcs yx_{n-1} and yx_3 forms a Hamiltonian bypass. This completes the discussion of induction first step for $(a = 1)$ Case I.

Now we consider the induction first step for Case II.

Case II. $a = 1$. Without loss of generality, we may assume that $y \to \{x_3, x_{n-1}\}$, $x_2y \in D$ and $d(y, \{x_1, x_4, x_{n-2}\}) = 0$. By induction first step of Case I, we may assume that y, x_5 also are non-adjacent. This together with $d(y) \leq n-2$, $x_2y \in D$ and the condition (*) implies that

$$d^+(x_2, \{x_4, x_5, x_{n-2}\}) = 0, \tag{5}$$

and hence, by Lemma 11, in particular, the vertices x_2, x_4 are non-adjacent. If $x_{n-2}x_1 \in D$, then the cycle $C' := x_{n-2}x_1x_2yx_3 \dots x_{n-2}$ has length $n-1$ missing the vertex x_{n-1} and $\{x_{n-2}, y\} \to x_{n-1} \to x_1$, i.e., for the cycle C' and vertex x_{n-1} the considered induction first step of Case I holds. Assume therefore that $x_{n-2}x_1 \notin D$. Then x_1, x_{n-2} are non-adjacent (Lemma 11). It is not difficult to see that x_1 has a partner on $C[x_3, x_{n-3}]$. Indeed, for otherwise from Lemma 5(i) it follows that $d(x_{n-1}) \geq n-1$ and hence by Lemma 2 and Remark (ii), we have

$$n - 1 \leq d(x_1) = d(x_1, \{x_2, x_{n-1}\}) + d(x_1, C[x_3, x_{n-3}]) \leq n - 2,$$

which is a contradiction. Thus, indeed x_1 has a partner on $C[x_3, x_{n-3}]$. Let the arc $x_kx_{k+1} \in C[x_3, x_{n-3}]$ be a partner of x_1. Note that $k \in [4, n-4]$ (by Lemma 11). Therefore, neither the vertex x_2 nor the arc x_1x_2 has a partner on $C[x_3, x_{n-1}]$ (for otherwise, by Multi-Insertion Lemma there is an (x_3, x_{n-1})-path with vertex set $V(C)$, which together with the arcs yx_3 and yx_{n-1} forms a Hamiltonian bypass). Recall that $a(x_2, x_4) = 0$ and $x_2x_5 \notin D$ (by (5). Now using Lemma 2(ii) and Remark (ii) we obtain that

$$d(x_2) = d(x_2, \{y, x_1, x_3\}) + d(x_2, C[x_5, x_{n-1}]) \leq n - 2.$$

This together with $x_1 \rightarrow \{x_2, x_{k+1}\}$ and the condition (*) implies that x_2 and x_{k+1} are adjacent. Then $x_{k+1}x_2 \in D$ (if $x_2x_{k+1} \in D$, then the arc x_1x_2 has a partner on $C[x_3, x_{n-1}]$). By a similar argument, we conclude that $\{x_{n-2}, x_{n-1}\} \rightarrow x_2$. Then $C' := x_{n-2}x_2yx_3x_4 \ldots x_kx_1x_{k+1} \ldots x_{n-2}$ is a cycle of length $n-1$, which does not contain the vertex x_{n-1} and $d(x_{n-1}, \{x_{n-2}, x_2, y\}) = 3$, a contradiction against Lemma 9 and hence, the discussion of case $a = 1$ is completed.

The induction hypothesis. Now we suppose that the theorem is true if D contains a cycle $C := x_1x_2 \ldots x_{n-1}x_1$ of length $n-1$ missing the vertex y for which there is a vertex x_i such that $d(y, \{x_{i+2}, x_{i+3}, \ldots, x_{i+j}\}) = 0$ and (i) $x_i \rightarrow y \rightarrow x_{i+1}$ and the vertices y, x_{i+j+1} are adjacent or (ii) $y \rightarrow \{x_{i+1}, x_{i+j+2}\}$ and $x_{i+j+1}y \in D$, where $2 \leq j \leq a \leq n-6$.

Before dealing with Cases I and II, it is convenient to prove the following general claim.

Claim. Let $C := x_1x_2 \ldots x_{n-1}x_1$ be an arbitrary cycle of length $n-1$ in D and let y be the vertex not on C and let $d(y, \{x_1, x_2, \ldots, x_a\}) = 0$, where $a \geq 2$.If (i) $x_{n-2}y, yx_{n-1} \in D$ and the vertices y and x_{a+1} are non-adjacent or (ii) $x_{a+1}y, yx_{a+2}$ and $yx_{n-1} \in D$, then $x_{k-1}x_{k+1} \notin D$ for all $k \in [1, a]$.

Proof of the Claim: Suppose, on the contrary, that $x_{k-1}x_{k+1} \in D$ for some $k \in [1, a]$, then $C' := x_{n-2}yx_{n-1}x_1 \ldots x_{k-1}x_{k+1} \ldots x_{n-2}$ or $C'' := x_{n-1}x_1 \ldots x_{k-1}x_{k+1} \ldots x_{a+1} yx_{a+2} \ldots x_{n-2}x_{n-1}$ is a cycle of length $n-1$ missing the vertex x_k for (i) and (ii), respectively. Therefore, $d(x_k) \leq n-2$ (by Remark (iv)). By the induction hypothesis x_k is not adjacent with vertices $x_{k+2}, x_{k+3}, \ldots, x_{k+a}, x_{k-2}, x_{k-3}, \ldots x_{k-a}$. In particular, $d^-(x_k, \{x_{k+1}, x_{k+2}, \ldots, x_{a+1}\}) = 0$ and $d^-(x_k, C[x_{n-2}, x_{k-2}]) = 0$ (it is easy to show that in both cases $x_{n-2}x_k \notin D$). Since $d(x_k) \leq n-2$ and $x_{k-2}x_k \notin D$, by Lemma 5(i), the vertex x_k has a partner on $C[x_{a+2}, x_{n-2}]$, say the arc $x_jx_{j+1} \in C[x_{a+2}, x_{n-2}]$ is a partner of x_k, i.e., $x_jx_k, x_kx_{j+1} \in D$. Therefore $x_{n-1}x_1 \ldots x_{k-1}x_{k+1} \ldots x_ax_{a+1} \ldots x_jx_kx_{j+1} \ldots x_{n-2}x_{n-1}$ is a cycle of length $n-1$ missing the vertex y, for which $d(y, C[x_1, x_a] - \{x_k\}) = 0$ and $x_{n-2}y, yx_{n-1} \in D$, x_{a+1}, y are non-adjacent or $yx_{n-1}, x_{a+1}y, yx_{a+2} \in D$ for (i) and (ii), respectively. Therefore, by the induction hypothesis D contains a Hamiltonian bypass, a contradiction to our assumption. The claim is proved. □ **Case I**. Without loss of generality, we may assume that $d(y, \{x_2, x_3, \ldots, x_{a+1}\}) = 0$, where $a \geq 2$, $x_{n-1}y, yx_1 \in D$ and the vertices y, x_{a+2} are non-adjacent.

Notice, the condition (*) implies that for all $i \in [2, a+1]$, $x_{n-1}x_i \notin D$, since $x_{n-1}y \in D$, $d(y) \leq n-2$ and the vertices x_i, y are non-adjacent.

Subcase I.1. There are integers k and l with $1 \leq l < k \leq a+2$ such that $x_kx_l \in D$. Without loss of generality, we assume that $k-l$ is as small as possible. From Remark (ii) and Lemma 11 it follows that $k-l \geq 3$. If every vertex $x_i \in C[x_{l+1}, x_{k-1}]$ has a partner on the path $P := x_kx_{k+1} \ldots x_{n-1}yx_1 \ldots x_l$, then by Multi-Insertion Lemma there exists an (x_k, x_l)-Hamiltonian path, which together with the arc x_kx_l forms a Hamiltonian bypass. Assume therefore that some vertex $x_i \in C[x_{l+1}, x_{k-1}]$ has no partner on P. From the minimality of $k-l \geq 3$ and Claim 1 it follows that $x_{i-2} \in C[x_l, x_k]$ and $a(x_i, x_{i-2}) = 0$ or $x_{i+2} \in C[x_l, x_k]$ and $a(x_i, x_{i+2}) = 0$. Therefore by the minimality of $k-l$ we have

$$d(x_i, C[x_l, x_k]) \leq k - l - 1. \tag{6}$$

Since x_i has no partner on the path $C[x_{k+1}, x_{n-1}]$, and if $l \geq 2$ also on $C[x_1, x_{l-1}]$, using Lemma 2 with the fact that $x_{n-1}x_i \notin D$ we obtain

$$d(x_i, C[x_{k+1}, x_{n-1}] \leq n - k - 1 \quad \text{and if} \quad l \geq 2, \quad \text{then} \quad d(x_i, C[x_1, x_{l-1}]) \leq l.$$

The last two inequalities together with (6) give: if $l \geq 2$, then $d(x_i) \leq n - 2$, and if $l = 1$, then $d(x_i) \leq n - 3$. Thus, $d(x_i) \leq n - 2$. In addition, Claim 1 and $x_{n-1}x_i \notin D$ imply that $x_{i-2}x_i \notin D$. Therefore, by Lemma 5(i), x_i has a partner on P, which is contrary to our assumption.

Subcase I.2. For any pair of integers k and l with $1 \leq l < k \leq a + 2$, $x_k x_l \notin D$. Then it is easy to see that for each $x_i \in C[x_2, x_{a+1}]$),

$$d(x_i, C[x_1, x_{a+2}]) \leq a, \tag{7}$$

since $x_{i-2} \in C[x_1, x_{a+2}]$ and $a(x_i, x_{i-2}) = 0$ or $x_{i+2} \in C[x_1, x_{a+2}]$ and $a(x_i, x_{i+2}) = 0$.

We first show that every vertex $x_i \in C[x_2, x_{a+1}]$ has a partner on $C[x_{a+3}, x_{n-1}]$. Assume that this is not the case, i.e., some vertex $x_i \in C[x_2, x_{a+1}]$ has no partner on $C[x_{a+3}, x_{n-1}]$. Then, since $x_{n-1}x_i \notin D$, by Lemma 2(ii) we have that $d(x_i, C[x_{a+3}, x_{n-1}]) \leq n - a - 3$. This inequality togeter with (7) gives $d(x_i) \leq n - 3$, a contradiction against Lemma 5(i), since $x_{i-2}x_i \notin D$.

Thus, each vertex $x_i \in C[x_2, x_{a+1}]$ has a partner on $C[x_{a+3}, x_{n-1}]$. Therefore, by Multi-Insertion Lemma there is an (x_{a+3}, x_{n-1})-path, say R, with vertex set $V(C) - \{x_1, x_{a+2}\}$. If $yx_{a+2} \in D$, then $[yx_1; yx_{a+2}Rx_1]$ is a Hamiltonian bypass. Assume therefore that $yx_{a+2} \notin D$. Then $x_{a+2}y \in D$. By Lemma 6(i) and by the induction hypothesis, we have $yx_{a+3} \in D$ and $d(y, \{x_{a+4}, x_{a+5}\}) = 0$. This together with $x_{a+2}y \in D$, $d(y) \leq n - 2$ and the condition (*) implies that

$$d^+(x_{a+2}, , \{x_{a+4}, x_{a+5}\}) = 0, \tag{8}$$

in particular, by Lemma 11, $a(x_{a+2}, a_{a+4}) = 0$. Since $yx_{a+3} \in D$ and each vertex $x_i \in C[x_2, x_{a+1}]$ has a partner on $C[x_{a+3}, x_{n-1}]$, to show that D contains a Hamiltonian bypass, by Multi-Insertion Lemma it suffices to prove that x_{a+2} also has a partner on $C[x_{a+3}, x_{n-1}]$. Assume that x_{a+2} has no partner on $C[x_{a+3}, x_{n-1}]$. Then, since the vertices x_a and x_{a+2} are non-adjacent (Claim 1 and Lemma 11), from Lemma 5(i) it follows that $d(x_{a+2}) \geq n - 1$. On the other hand, using (7), (8), $d(x_{a+2}, \{x_a, x_{a+4}\}) = 0$ and Lemma 2, we obtain

$$n - 1 \leq d(x_{a+2}) = d(x_{a+2}, C[x_1, x_{a+1}]) + d(x_{a+2}, \{y, x_{a+3}) + d(x_{a+2}, C[x_{a+5}, x_{n-1}]) \leq n - 3,$$

a contradiction. So, x_{a+2} also has a partner on $C[x_{a+3}, x_{n-1}]$ and the discussion of Case I is completed.

Case II. Without loss of generality, we assume that $d(y, \{x_1, x_2, \ldots, x_a\}) = 0$, where $a \geq 2$, $x_{a+1}y \in D$ and $y \to \{x_{n-1}x_{a+2}\}$.

By the considered Case I, without loss of generality, we may assume that $d(y, \{x_{n-2}, x_{a+3}, x_{a+4}\}) = 0$. Since $x_{a+1}y \in D$, $d(y) \leq n - 2$ and $d(y, \{x_1, x_2, \ldots, x_a, x_{a+3}, x_{a+4}, x_{n-2}\}) = 0$, the condition (*) implies that

$$d^+(x_{a+1}, \{x_1, x_2, \ldots, x_a, x_{a+3}, x_{a+4}, x_{n-2}\}) = 0. \tag{9}$$

Subcase II.1. There are integers k and l with $1 \leq l < k \leq a + 1$ such that $x_k x_l \in D$. By (9), $k \neq a + 1$. Without loss of generality, we assume that $k - l$ is as small as possible. By Remark (ii) and Lemma 11 we have $k - l \geq 3$.

We first show that each vertex of $C[x_{l+1}, x_{k-1}]$ has a partner on the path $P := x_k x_{k+1} \dots x_{a+1} y \ x_{a+2} \dots x_{n-1} x_1 \dots x_l$. Assume that this is not the case and let $x_i \in C[x_{l+1}, x_{k-1}]$ have no partner on P. Then, since $x_{i-2}x_i \notin D$ (Claim 1), from Lemma 5(i) and the minimality of $k-l$ it follows that $d(x_i) \geq n-1$. On the other hand, using the minimality of $k-l$ and the fact that $x_{i-2} \in C[x_l, x_k])$ and $a(x_i, x_{i-2}) = 0$ or $x_{i+2} \in C[x_l, x_k]$ and $a(x_i, x_{i+2}) = 0$ we obtain

$$d(x_i, C[x_l, x_k]) \leq k - l - 1.$$

In addition, by Lemma 2 and $x_{a+1}x_i \notin D$ we also have

$$d(x_i, C[x_{k+1}, x_{a+1}]) \leq a - k + 1 \quad \text{and} \quad d(x_i, C[x_{a+2}, x_{l-1}]) \leq n - a + l - 2.$$

Summing the last three inequalities gives $d(x_i) \leq n-2$, which contradicts that $d(x_i) \geq n-1$. Thus, indeed each vertex $x_i \in C[x_{l+1}, x_{k-1}]$ has a partner on P. Then by Multi-Insertion Lemma there is an (x_k, x_l)-Hamiltonian path , which together with the arc $x_k x_l$ forms a Hamiltonian bypass.

Subcase II.2. There are no i and j such that $1 \leq i < j \leq a+1$ and $x_j x_i \notin D$. If every vertex $x_i \in C[x_1, x_{a+1}])$ has a partner on $C[x_{a+2}, x_{n-1}]$, then by Multi-Insertion Lemma there is an (x_{a+2}, x_{n-1})-path, say R, with vertex set $V(C)$. Therefore $[yx_{n-1}; yR]$ is a Hamiltonian bypass. Assume therefore that there is a vertex $x_i \in C[x_1, x_{a+1}]$ which has no partner on $C[x_{a+2}, x_{n-1}]$.

Let $x_{i-2}x_i \notin D$, then from Lemma 5(i) it follows that $d(x_i) \geq n-1$.

Assume first that $d(x_i, C[x_1, x_{a+1}]) = a-1$. Using Lemma 2 we obtain that if $x_i \neq x_{a+1}$, then

$$n - 1 \leq d(x_i) = d(x_i, C[x_1, x_{a+1}]) + d(x_i, C[x_{a+2}, x_{n-1}]) \leq n - 2,$$

and, since $x_{a+1}x_{a+3} \notin D$, if $x_i = x_{a+1}$, then

$$n - 1 \leq d(x_{a+1}) = d(x_{a+1}, C[x_1, x_{a+1}]) + d(x_{a+1}, \{y, x_{a+2}) + d(x_{a+1}, C[x_{a+3}, x_{n-1}]) \leq n - 2,$$

a contradiction.

Assume second that $d(x_i, C[x_1, x_{a+1}]) = a$. Then from Claim 1 and Lemma 11 it follows that $a = 2$, $x_i = x_2$, $d(x_2, \{x_1, x_3\}) = 2$ and $d(x_2, \{x_{n-1}\}) = 0$. Then

$$n - 1 \leq d(x_2) = d(x_2, \{x_1, x_3\}) + d(x_2, C[x_4, x_{n-2}]) \leq n - 2,$$

a contradiction.

Let now $x_{i-2}x_i \in D$. Then, by Claim 1, $x_i = x_1$ and $x_{n-2}x_1 \in D$. We consider the cycle $C' := x_{n-3}x_{n-2}x_1x_2 \dots x_a x_{a+1} y x_{a+2} \dots x_{n-3}$ of length $n-1$ missing the vertex x_{n-1}. Then $\{x_{n-2}, y\} \rightarrow x_{n-1}$ and $x_{n-1}x_1 \in D$, i.e., for the cycle C' and the vertex x_{n-1} Case I holds, since $|\{x_2, x_3, \dots, x_{a+1}\}| = a$. The discussion of Case II is completed and with it the proof of the theorem is also completed. □

5. Concluding Remarks

The following two examples of digraphs show that if the minimal semi-degree of a digraph is equal to one, then the theorem is not true:

(i) Let $D(7)$ be a digraph with vertex set $\{x_1, x_2, \dots, x_6, y\}$ and let $x_1x_2 \dots x_6x_1$ be a cycle of length 6 in $D(7)$. Moreover, $N^+(y) = \{x_1, x_3, x_5\}$, $N^-(y) = \{x_2, x_4, x_6\}$, $x_1x_3, x_3x_5, x_5x_1 \in$

$D(7)$ and $D(7)$ has no other arcs. Note that $d^-(x_2) = d^-(x_4) = d^-(x_6) = 1$ and $D(7)$ contains no dominated pair of non-adjacent vertices. It is not difficult to check that $D(7)$ contains no Hamiltonian bypass.

(ii) Let $D(n)$ be a digraph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and let $x_1x_2 \ldots x_nx_1$ be a Hamiltonian cycle in $D(n)$. Moreover, $D(n)$ also contains the arcs $x_1x_3, x_3x_5, \ldots, x_{n-2}x_n$ (or $x_1x_3, x_3x_5, \ldots, x_{n-3}x_{n-1}$, $x_{n-1}x_1$ and $D(n)$ has no other arcs. Note that $D(n)$ contains no dominated pair of non-adjacent vertices, $d^-(x_2) = d^+(x_2) = 1$. It is not difficult to check that $D(n)$ contains no Hamiltonian bypass.

We believe that Theorem 12 also is true if we require that the minimum in-degree at least two, instead of three.

In [2] and [3] Theorem 13 and Theorem 14 were proved, respectively.

Theorem 13:*(Bang-Jensen, Gutin, H. Li [2]). Let D be a strong digraph of order $n \geq 3$. Suppose that $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Theorem 14:*(Bang-Jensen, Guo, Yeo [3]). Let D be a strong digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n - 1$ and $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Note that Theorem 14 generalizes Theorem 13.

In [9] and [10] the following results were proved:

Theorem 15:*([9]). Let D be a strong digraph of order $n \geq 4$ which is not a directed cycle. Suppose that $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour. Then either D contains a pre-Hamiltonian cycle or n is even and $D = K^*_{n/2,n/2}$.*

Theorem 16: *([10]). Let D be a strong digraph of order $n \geq 4$ which is not a directed cycle. Suppose that $d(x) + d(y) \geq 2n - 1$ and $min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$ for any pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour. Then D contains a pre-Hamiltonian cycle or a cycle of length $n - 2$.*

In view of Theorems 13-16, we pose the following problem:

Problem: Characterize those digraphs which satisfy the condition of Theorem 13 or 14 but have no Hamiltonian bypass.

References

[1] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, 2000.

[2] J. Bang-Jensen, G. Gutin, H. Li, “Sufficient conditions for a digraph to be Hamiltonian”, *Journal of Graph Theory*, vol. 22 no. 2, pp. 181–187, 1996.

[3] J. Bang-Jensen, Y. Guo, A.Yeo, “A new sufficient condition for a digraph to be Hamiltonian”, *Discrete Applied Mathematics*, vol. 95, pp. 77–87, 1999.

[4] A.Benhocine, “On the existence of a specified cycles in digraphs with constraints on degrees”, *Journal of Graph Theory*, vol. 8, pp.101-107, 1984.

[5] J.A. Bondy, C. Thomassen, “A short proof of Meyniel’s theorem”, *Discrete Mathematics*, vol. 19, no. 1, pp. 85-92, 1977.

[6] S.Kh. Darbinyan, ”A sufficient condition for the Hamiltonian property of digraphs with

large semidegrees", *Akademy Nauk Armyan. SSR Doklady*, (see also arXiv: 1111.1843v1 [math.CO] 8 Nov 2011), vol. 82, no. 1, pp. 6-8, 1986.

[7] S.Kh. Darbinyan, "On Hamiltonian bypasses in digraphs satisfying Meyniel-like conditions", *Mathematical Problems in Computer Science*, (see also Proceedings of 5-th science-technical conference for yang researchers, p. 23, Tsaghkadzor, Armenia, 1986), (in Russian), vol. 20, pp.7-19, 1998.

[8] S.Kh. Darbinyan, "On the specified cycles in oriented graphs", *Akademy Nauk Armyan. SSR Doklady*, (in Russian), vol. 84, no. 1, pp. 51–55, 1987.

[9] S.Kh. Darbinyan, I. A. Karapetyan, "On longest non-Hamiltonian cycles in digraphs with the conditions of Bang-Jensen, Gutin and Li", *Preprint available at htte: arXiv 1207.5643v2 [math.CO]*, 20 Sep 2012.

[10] S.Kh. Darbinyan, I. A. Karapetyan, "A note on long non-Hamiltonian cycles in one class of digraphs", *Preprint available at htte: arXiv 1209.4456v1 [math.CO]*, 20 Sep 2012.

[11] R. Häggkvist, C. Thomassen, "On pancyclic digraphs", *Journal Combinatorial Theory Ser. B*, vol. 20, pp. 20–40, 1976.

[12] A. Ghouila-Houri, "Une condition suffisante d'existence d'un circuit hamiltonien", *Comptes Rendus de I'Academie des Sciences Paris Ser. A-B*, no. 251, pp. 495–497, 1960.

[13] M. Meyniel, "Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe oriente", *Journal Combinatorial Theory Ser. B*, vol. 14, pp. 137-147, 1973.

[14] C.St.J.A. Nash-Williams, "Hamilton circuits in graphs and digraphsit. The many facts of graph theory", *Springer Lecture Notes.* 110, pp.237-243, 1969.

[15] C. Thomassen, "Long cycles in digraphs", *Proceeding London Mathematical Society*, vol. 3, no. 42, pp. 231–251, 1981.

[16] D. R. Woodall, "Sufficient conditions for circuits in graphs", *Proceeding London Mathematical Society*, no. 24, pp. 739–755, 1972.

Submitted 20.12.2013, accepted 28.02.2014.

Կողմնորոշված համիլտոնյան գրաֆների մի դասի համիլտոնյան շրջանցումների մասին

Ս. Դարբինյան և Ի. Կարապետյան

Ամփոփում

Կողմնորոշված գրաֆի համիլտոնյան շրջանցումը այդ գրաֆի մի ենթագրաֆ է, որը ստացվում է համիլտոնյան ցիկլի մեկ աղեղի կողմնորոշումը շրջելուց հետո: Ներկա աշխատանքում ապացուցվում է, որ եթե կողմնորոշված գրաֆը բավարարում է համիլտոնյանության մի հայտնի պայմանի (J.of Graph Theory 22(2) (1996) 181-187), և նրա գագաթների փոքրագույն մտնող և դուրսեկող աստիճանները փոքր չեն համապատասխանաբար,երեքից և երկուսից, ապա այդ գրաֆը պարունակում է համիլտոնյան շրջանցում:

О гамильтоновых обходах в одном классе гамильтоновых орграфов

С. Дарбинян и И. Карапетян

Аннотация

Доказывается, что любой сильно связный n-вершинный ($n > 3$) орграф, который удовлетворяет одному достаточному условию гамильтоновости орграфов (J.of Graph Theory 22(2) (1996) 181-187) и имеет минимальную полустепень исхода и захода не меньше чем 2 и 3, соответственно, содежит гамильтоновый обход, т.е., контур, который получается из гамильтонового контура после переориентации одной дуги.