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Method of Local Interchange for the Investigation of Gossip Problems: part 2

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Abstract

The method of construction of Gossip graphs providing a full information exchange with minimal number of calls in minimum time is described. The basis for the graphs of the presented class is the subgraph of canonical form obtained from NOHO graphs by applying the operation of local interchange on them developed by us in [19].

Keywords: Graphs, Networks, Telephone problem, Gossip problem.

1. Introduction

Gossiping is one of the basic problems of information dissemination in communication networks. The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see e.g. [1] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). Consider a set of n persons (nodes) each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with minimum length (minimal gossip scheme), by which all the nodes will know all pieces of a information (complete gossiping). It has been shown in numerous works [1]–[4] that the minimal number of calls is $2n-4$ when $n \geq 4$ and 1, 3 for $n = 2, 3$, respectively. Since then many variations of gossip problem have been introduced and investigated (see e.g. [5]–[15]).

Another variant of Gossip problem can be formulated by considering the minimum amount of time required to complete gossiping among n persons, where the calls between the non-overlapping pairs of nodes can take place simultaneously and each call requires one unit of time ([16]–[18]).

Obviously, the gossip problem can be easily modeled as a graph, whose vertices represent people in gossip scheme and edges represent calls between them (each of them has weight which represents the moment when communication took place). So the graph is called a complete gossip graph if there are ascending paths between all pairs of vertices of this graph.

In this paper we continue to consider the methods of local interchange on gossip graphs which are particular cases of operations defined in [1] and are defined and described in [19]. In section 2 we present a new use case of these operations concerning the maximum number of vertices in minimum gossip schemes with a minimum gossiping time, also describe the variations of gossip scheme depending on the size of basis.

2. Maximum number of vertices in minimum gossip graph with minimum gossiping time

In this section we are going to introduce a new use case of operations of local interchange which are described in [19]. It helps us to construct a gossip scheme with a minimum number of edges and minimum gossiping time (w.l.o.g. minimum gossip scheme/graph). Particularly, we will use them to obtain a canonical form of basis for minimum gossip scheme.

In [20] the class of NOHO graphs was described, which were gossip graphs with n vertices and $2n-4$ calls, where no one from the participants of gossip process listens to its own information. The minimum gossip graphs based on NOHO graphs were constructed first in [20]. In [21], this problem was considered and a wrong inference was made about the structure of the minimum gossip scheme. According to it, all minimum gossip schemes consist of a cube and eight minimum broadcast trees attached to its corner points. Later in [8], this statement was negated and it was shown that it was the only particular case of the structure of minimum gossip graphs. In general, the inner kernel of the gossip graph with $2n-4$ calls, which were presented in [8] as a poset of the edges of graph and represents their relations (lateral graph), could be isomorphic to cube, "twisted" cube and p-grid-kernel. In the current section we are going to present different ways of construction of the minimum gossip schemes. In [21] and [8] it was shown that the time required to perform gossiping with minimum number of calls and minimum time (in other words - with minimum number of rounds) is at least $2\lceil\log_2 n\rceil - 3$. The methods that allow to construct minimum gossip schemes were also considered that have an underlying basis (a minimum gossip graph) for gossiping, and new vertices participate in gossiping by forming the attached trees which are connected to basis. It is obvious, that if the basis has a minimum number of edges, then the full graph also would have the same. Hence, the main problem here is to perform gossiping in minimum time. In other words, the time in which the basis performs gossiping should be minimal and the corresponding attached trees should not affect it. So, our goal is to construct such gossip schemes with variety of size of the basis.

As was mentioned above, we will start from the application of operation A^+ defined in [19] on the NOHO graphs. Here are some definitions from [19].

Definition 1: *The permute higher operation $P^+(e)$ on a selected edge $e \in E(G)$ connecting vertices u and v is called a modification of G, which moves the edges of G adjacent to e as follows:*

$$E_u(P^+(e)G) = \rho_u^-(e,G) \cup \rho_v^+(e,G), \tag{1}$$

and

$$E_v(P^+(e)G) = \rho_v^-(e,G) \cup \rho_u^+(e,G). \tag{2}$$

Correspondingly, the permute lower operation $P^-(e)$ moves the edges of G adjacent to e as follows:

$$E_u(P^-(e)G) = \rho_u^+(e,G) \cup \rho_v^-(e,G), \tag{3}$$

and

$$E_v(P^-(e)G) = \rho_v^+(e,G) \cup \rho_u^-(e,G). \tag{4}$$

Definition 2: *Let us define an operation on gossip graphs* $A^{+}(e_1, e_2, \ldots, e_p) = (P^{+}(e_1)P^{+}(e_2)\ldots P^{+}(e_p))$ $(A^{-}(e_1, e_2, \ldots, e_p) = (P^{-}(e_1)P^{-}(e_2)\ldots P^{-}(e_p)))$ *is the sequence of permute higher (lower) operations on edges* e_i, $i = 1, \ldots, p$.

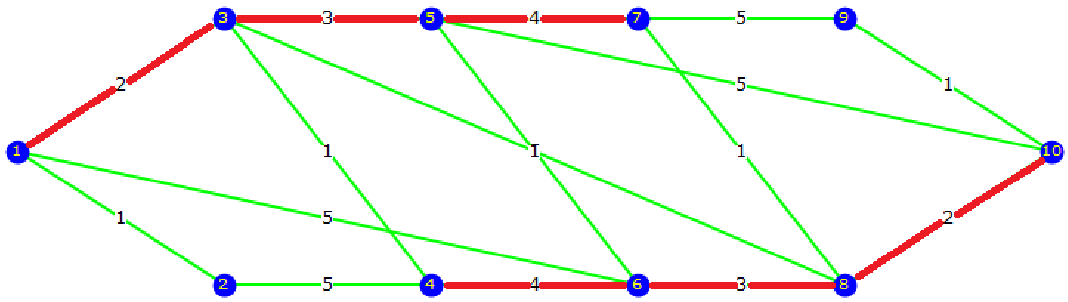

Fig. 1. NOHO graph.

Application of operations $A^{+}(e_1, e_2, \ldots, e_p)$ and $A^{+}(e'_1, e'_2, \ldots, e'_p)$ on NOHO graphs (Fig. 1) gives as new "canonical" form of minimum gossip graph, which is a more convenient form of basis (Fig 2). Here $p = 3$ and the edges e_1, e_2, e_3 (e'_1, e'_2, e'_3) are highlighted in red.

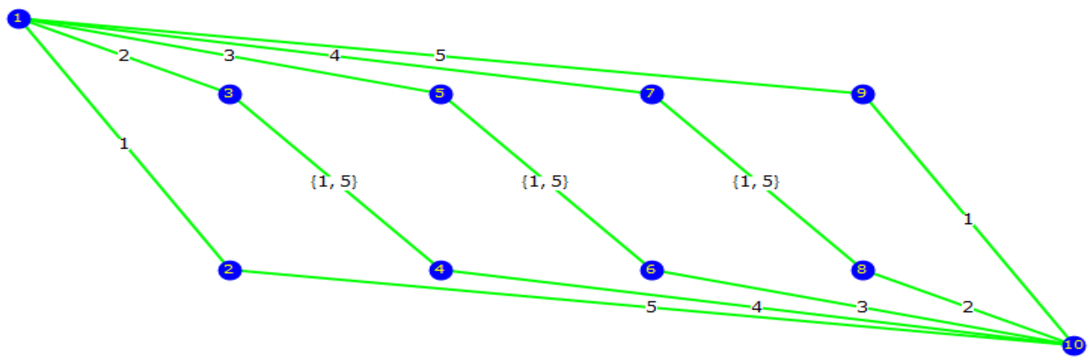

Fig. 2. Minimum gossip graph in "canonical" form.

After this transformation the obtained graph will be used as a basis with size $k = n'/2$, where n' is the number of vertices of the basis. Now consider the new set of vertices with the same size that communicates with the given set by incoming and outgoing calls. Let the number of rounds by which such a communication takes place be denoted by r. Obviously, the number of rounds to perform gossiping will be at least $2r + k$. This estimate will be reached only if the attached trees would not affect gossiping process of basis. It means, that the process of involving new vertices in gossiping should be in parallel with the main gossiping process of the basis. Hence, we have to modify the form of the basis such that we could make in-calls and out-calls with the attached trees in process of gossiping of the basis. For that, we should reorder the calls of basis, in such a way that it will be possible to make such in-calls and out-calls. In other words the calls between two parts of gossip scheme should have an increasing and decreasing order. An additional note is that depending on r the number of attached vertices increases exponentially. Such gossiping process for $n = 24$ is demonstrated in Fig. 3, where $r = 2$ and $k = 3$.

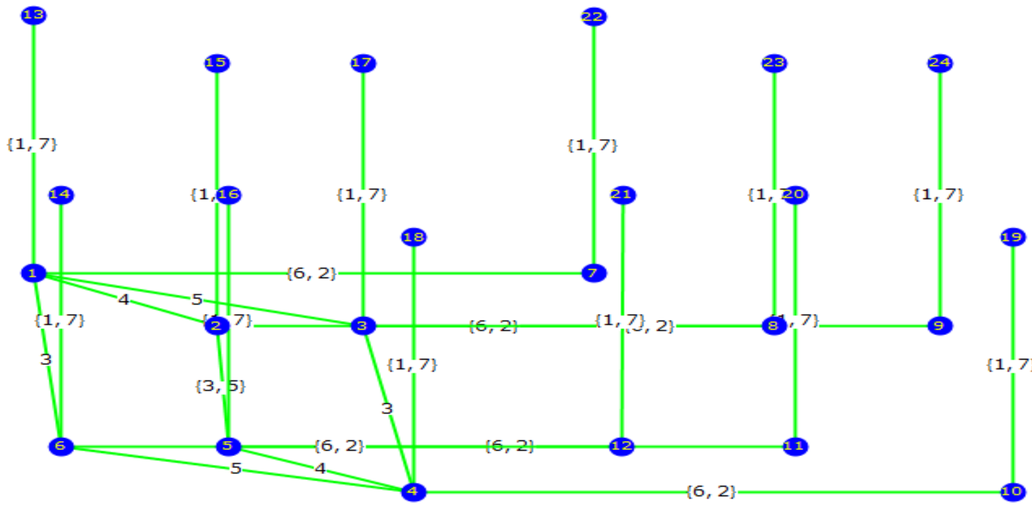

Fig. 3. Minimum gossip graph ($n = 24, k = 3, r = 2$).

Obviously, the number of rounds required to perform gossiping is $T' = 2r + k$ which coincides with the minimum possible number of rounds for some values of n, but this scheme is not the only possible one to perform gossiping in minimum time. By changing the size of the basis (k) and number of rounds of the "batch" calls (r) a new minimum gossiping scheme will be obtained (see Fig. 4).

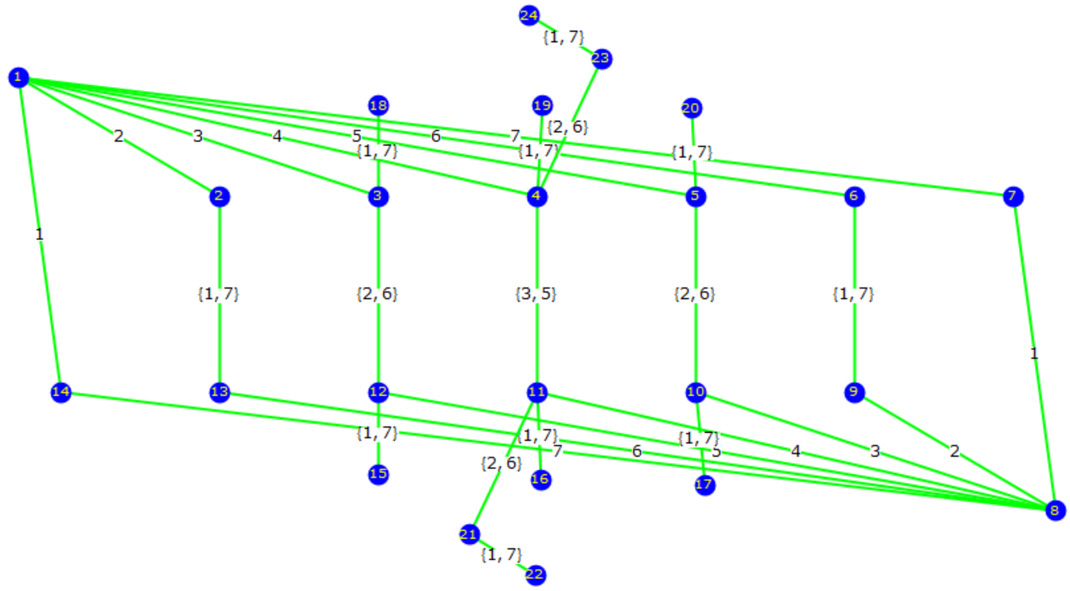

Fig. 4. Minimum gossip graph ($n = 24, k = 7, r = 0$).

From these examples it is obvious that in each round new vertices are included in communication, but the number of these vertices is limited. So, taking these into account the following lemma could be formulated.

Lemma 1: *The number of vertices in minimum gossip graph with the size of basis k and the number of rounds of the "batch" calls r is limited by the following estimates:*

$$n \leq 2^{k/2+r+1}, \text{ if } k \text{ is even}, \tag{5}$$

$$n \leq 3 \times 2^{(k-1)/2+r}, \text{ if } k \text{ is odd}. \tag{6}$$

Proof. To prove these estimates let us consider the process of gossiping more detailed. The number of vertices that could be involved in gossiping consists of two components n_1 and n_2, where n_1 is the number of vertices which are involved in gossiping process due to "batch" calls and n_2 is the amount of vertices involved in gossiping through the attached trees. The dependency between the number of rounds of the "batch" calls r and the n_1 number of vertices is obvious - $n_1 \leq 2^r \times 2k$. On the other side, it is easy to note that the number of vertices involved in the attached trees is $n_2 \leq 2^r \times (2 \sum_{i=2}^{\lceil \frac{k}{2} \rceil - 1} (k-2i)2^{i-2})$ and it also depends on r in exponential rule. Altogether, by considering that $n = n_1 + n_2$ for the number of vertices of the minimum gossip graph we get the following estimate:

$$n \leq 2^r (2k + 2 \sum_{i=2}^{\lceil \frac{k}{2} \rceil - 1} (k-2i)2^{i-2}). \tag{7}$$

So, from this expression the estimates mentioned in lemma immediately follow. ■

However, in case of even k it is possible by changing the structure and gossiping time of basis to reach more vertices involved in gossiping. The example given in Fig. 5 demonstrates this for $k = 8$. According to lemma the number of vertices in this case is $n \leq 32$, but we increase the number of rounds of basis by one and change the direction of the "middle" calls and it gives us the opportunity to involve 8 new vertices in gossiping. In fact, our modification has changed the structure of the standard p-grid-kernel to one of it's linear extension (see. [8]) and the number of new vertices involved in gossiping process after this modification is $2^{\frac{k}{2}+r-1}$.

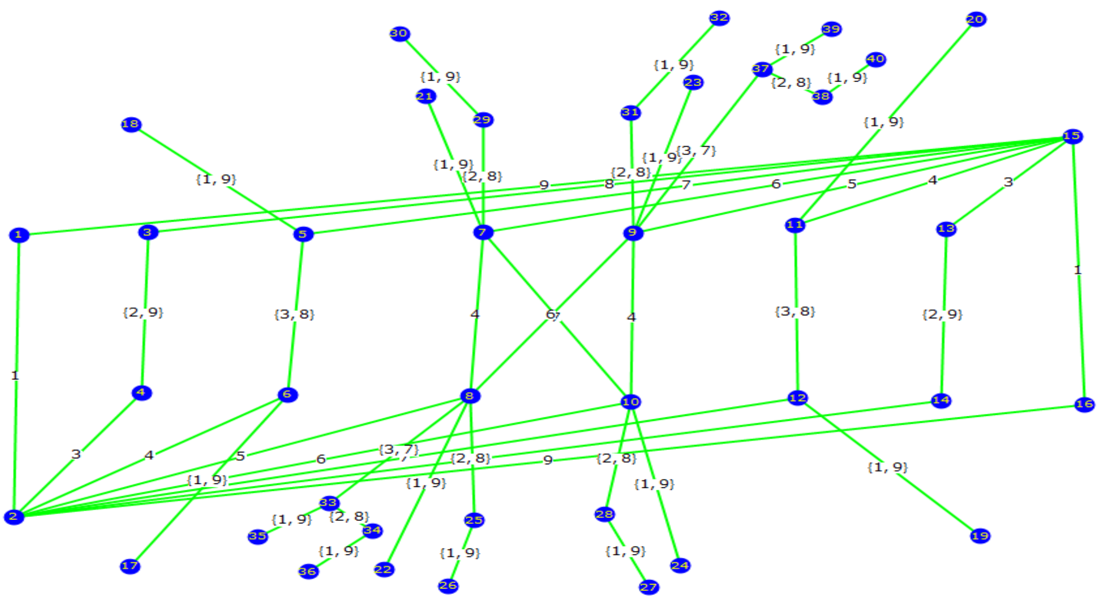

Fig. 5. Minimum gossip graph ($n = 40, k = 8, r = 0$).

Now let us consider the number of rounds which are necessary to perform gossiping with minimum number of calls. Let it be denoted by T. As it was already mentioned $T = 2\lceil \log_2 n \rceil - 3$. Our goal is to show the dependency between T and n depending on the construction of gossiping scheme. In other words, we offer the method of construction of minimum gossip graphs for the particular values of n, where the values of k and r can vary. Let T' be the minimum number of rounds of the gossip scheme with $2n-4$ calls, the size of the basis k and the number of rounds of "batch" calls r.

Theorem 1: *In gossiping scheme with the size of the basis k and the number of rounds of the "batch" calls r the number of rounds necessary to complete gossiping among n nodes is:*

$$T' = 2\lceil \log_2 n \rceil - 2, \text{ if } k \text{ is even}, \tag{8}$$

$$T' = 2\left\lceil \log_2 \frac{n}{3} \right\rceil + 1, \text{ if } k \text{ is odd}. \tag{9}$$

Proof. In case of even k we have $n \leq 2^{\frac{k}{2}+r+1} = 2^{\frac{T'}{2}+1}$, since $T' = 2r + k$, it follows that when n has the form as above then $T' = 2\lceil \log_2 n \rceil - 2$. When k is odd $n \leq 3 \times 2^{\frac{k-1}{2}+r} = 3 \times 2^{\frac{T'-1}{2}}$, hence $T' = 2\left\lceil \log_2 \frac{n}{3} \right\rceil + 1$. Moreover, by considering the "extended" values of n obtained after modification described above (when it is even) we have $n \leq 2^{\frac{k}{2}+r-1} + 2^{\frac{k}{2}+r-1}$. Since, here $T' = k + 1 + 2r$ and $T' = 2\left\lceil \log_2 \frac{n}{5} \right\rceil + 3$ follows immediately. It is obvious, that for some values of n T' coincides with minimum possible number of rounds $2\lceil \log_2 n \rceil - 3$. Hence, in these ranges of n the values of k and r can vary and give different construction options of minimum gossip graphs. In Fig. 6 some of these ranges are demonstrated, the construction is minimum in the highlighted ranges.

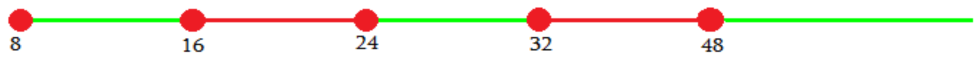

Fig. 6. Ranges in which the construction is minimal

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3. Conclusion

As a conclusion we can say, that in this paper the method of construction of minimum gossip graphs was presented, based on "canonical" form basis graph with size k, which is in turn obtained from NOHO graphs by applying operations of local interchange.

Assuming that the calls between non-overlapping pairs of nodes can take place simultaneously, the minimum amount of time $T(n)$ required to complete gossiping is $\lceil \log_2 n \rceil$ for even n and $\lceil \log_2 n \rceil + 1$ for odd n. So, our further studies concern to the "reverse" problem of finding the minimum number of calls of a gossip scheme with time (rounds) $T(n)$.

We also find it interesting to use the operations of local interchange in fault-tolerant broadcast schemes when at most k line faults are possible in the communication process (see [24, 25]). It can be a valuable tool in order to determine more tie estimates for $B_k(n)$ - the number of minimum necessary lines to perform k-fault-tolerant broadcasting.

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Gossip խնդիրների հետազոտումը "Լոկալ փոխանակման" մեթոդի միջոցով. մաս 2

Վ. Հովնանյան, Ս. Պողոսյան և Վ. Պողոսյան

Ամփոփում

Նկարագրված է մինիմալ զանգերով մինիմալ ժամանակում ինֆորմացիոն լրիվ փոխանակում ապահովող Gossip գրաֆների որոշակի դասի կառուցման եղանակ: Ներկայացվող դասի գրաֆների համար, որպես բազային ենթագրաֆ են հանդիսացել NOHO գրաֆների վրա [19]-ում մեր կողմից մշակված "Լոկալ փոխանակման" մեթոդի միջոցով կանոնիկ տեսքի բերված ենթագրաֆները:

Исследование Gossip задач методом "Локального обмена": часть 2

В.Овнанян, С. Погосян и В. Погосян

Аннотация

Описан метод построения определенного класса Gossip графов, обеспечивающих полный информационый обмен с помощью минимального числа звонков за минимальное время. Для представленного класса графов, базовыми подграфами являются графы канонического вида, полученные путем преобразования NOHO графов с помощью разработанного в [19] нами метода Локального обмена.