Interval Total Colorings of Complete Multipartite Graphs and Hypercubes

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Abstract

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. An interval total t-coloring of a graph G is a total coloring of G with colors $1, \ldots, t$ such that all colors are used, and the edges incident to each vertex v together with v are colored by $d_G(v) + 1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. In this paper we prove that all complete multipartite graphs with the same number of vertices in each part are interval total colorable. Moreover, we also give some bounds for the minimum and the maximum span in interval total colorings of these graphs. Next, we investigate interval total colorings of hypercubes Q_n. In particular, we prove that Q_n ($n \geq 3$) has an interval total t-coloring if and only if $n + 1 \leq t \leq \frac{(n+1)(n+2)}{2}$.

Keywords: Total coloring, Interval total coloring, Interval coloring, Complete multipartite graph, Hypercube.

1. Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. Let $VE(G)$ denote the set $V(G) \cup E(G)$. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$ and the total chromatic number of G by $\chi''(G)$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. For $F \subseteq E(G)$, the subgraph obtained by deleting the edges of F from G is denoted by $G - F$. The terms and concepts that we do not define can be found in [1, 20, 21].

Let $\lfloor a \rfloor$ denote the largest integer less than or equal to a. For two positive integers a and b with $a \leq b$, the set $\{a, \ldots, b\}$ is denoted by $[a, b]$ and called an interval. For an interval $[a, b]$ and a nonnegative number p, the notation $[a, b] \oplus p$ means $[a + p, b + p]$.

A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors appearing on edges incident to v. A proper edge-coloring of a graph G is an interval t-coloring [2] if all colors are used, and for any $v \in V(G)$, the set $S(v, \alpha)$ is an interval of integers. A graph G is interval colorable if it has an interval

t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathcal{N}$. For a graph $G \in \mathcal{N}$, the least (the minimum span) and the greatest (the maximum span) values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. If α is a total coloring of a graph G, then $S[v, \alpha]$ denotes the set $S(v, \alpha) \cup \{\alpha(v)\}$.

A graph $K_{n_1, \ldots, n_r}$ is a complete r-partite ($r \geq 2$) graph if its vertices can be partitioned into r independent sets $V_1, \ldots, V_r$ with $|V_i| = n_i$ ($1 \leq i \leq r$) such that each vertex in V_i is adjacent to all the other vertices in V_j for $1 \leq i < j \leq r$. A complete r-partite graph $K_{n, \ldots, n}$ is a complete balanced r-partite graph if $|V_1| = |V_2| = \cdots = |V_r| = n$. Clearly, if $K_{n, \ldots, n}$ is a complete balanced r-partite graph with n vertices in each part, then $\Delta(K_{n, \ldots, n}) = (r - 1)n$. Note that the complete graph K_n and the complete balanced bipartite graph $K_{n,n}$ are special cases of the complete balanced r-partite graph. The total chromatic numbers of complete and complete bipartite graphs were determined by Behzad, Chartrand and Cooper [3]. They proved the following theorems:

Theorem 1: *For any $n \in \mathbf{N}$, we have*

$$\chi''(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n + 1, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2: *For any $m, n \in \mathbf{N}$, we have*

$$\chi''(K_{m,n}) = \begin{cases} \max\{m, n\} + 1, & \text{if } m \neq n, \\ n + 2, & \text{if } m = n. \end{cases}$$

A more general result on total chromatic numbers of complete balanced multipartite graphs was obtained by Bermond [4].

Theorem 3: *For any complete balanced r-partite graph $K_{n, \ldots, n}$ ($r \geq 2, n \in \mathbf{N}$), we have*

$$\chi''(K_{n, \ldots, n}) = \begin{cases} (r - 1)n + 2, & \text{if } r = 2 \text{ or } r \text{ is even and } n \text{ is odd,} \\ (r - 1)n + 1, & \text{otherwise.} \end{cases}$$

An interval total t-coloring of a graph G is a total coloring α of G with colors $1, \ldots, t$ such that all colors are used, and for any $v \in V(G)$, the set $S[v, \alpha]$ is an interval of integers. A graph G is interval total colorable if it has an interval total t-coloring for some positive integer t. The set of all interval total colorable graphs is denoted by $\mathcal{T}$. For a graph $G \in \mathcal{T}$, the least (the minimum span) and the greatest (the maximum span) values of t for which G has an interval total t-coloring are denoted by $w_\tau(G)$ and $W_\tau(G)$, respectively. Clearly,

$$\chi''(G) \leq w_\tau(G) \leq W_\tau(G) \leq |V(G)| + |E(G)| \text{ for every graph } G \in \mathcal{T}.$$

The concept of interval total coloring was introduced by Petrosyan [5]. In [5, 6], the author proved that if $m + n + 2 - \gcd(m, n) \leq t \leq m + n + 1$, then the complete bipartite graph $K_{m,n}$ has an interval total t-coloring. Later, Petrosyan and Torosyan [18] obtained the following results:

Theorem 4: *For any $m, n \in \mathbf{N}$, we have*

$$W_\tau(K_{m,n}) = \begin{cases} m + n + 1, & \text{if } m = n = 1, \\ m + n + 2, & \text{otherwise.} \end{cases}$$

Theorem 5: *For any $n \in \mathbf{N}$, we have*

(1) $K_{n,n} \in \mathcal{T}$,

(2) $w_\tau(K_{n,n}) = \chi''(K_{n,n}) = n + 2$,

(3) $W_\tau(K_{n,n}) = \begin{cases} 2n + 1, & \text{if } n = 1, \\ 2n + 2, & \text{if } n \geq 2, \end{cases}$

(4) *if $w_\tau(K_{n,n}) \leq t \leq W_\tau(K_{n,n})$, then $K_{n,n}$ has an interval total t-coloring.*

In [6], Petrosyan investigated interval total colorings of complete graphs and hypercubes, where he proved the following two theorems:

Theorem 6: *For any $n \in \mathbf{N}$, we have*

(1) $K_n \in \mathcal{T}$,

(2) $w_\tau(K_n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \frac{3}{2}n, & \text{if } n \text{ is even,} \end{cases}$

(3) $W_\tau(K_n) = 2n - 1$.

Theorem 7: *For any $n \in \mathbf{N}$, we have*

(1) $Q_n \in \mathcal{T}$,

(2) $w_\tau(Q_n) = \chi''(Q_n) = \begin{cases} n + 2, & \text{if } n \leq 2, \\ n + 1, & \text{if } n \geq 3, \end{cases}$

(3) $W_\tau(Q_n) \geq \frac{(n+1)(n+2)}{2}$,

(4) *if $w_\tau(Q_n) \leq t \leq \frac{(n+1)(n+2)}{2}$, then Q_n has an interval total t-coloring.*

Later, Petrosyan and Torosyan [18] showed that if $w_\tau(K_n) \leq t \leq W_\tau(K_n)$, then the complete graph K_n has an interval total t-coloring. In [13], Petrosyan and Shashikyan proved that trees have an interval total coloring. In [14, 15, 16], Petrosyan and Shashikyan investigated interval total colorings of bipartite graphs. In particular, they proved that regular bipartite graphs, subcubic bipartite graphs, doubly convex bipartite graphs, $(2, b)$-biregular bipartite graphs and some classes of bipartite graphs with maximum degree 4 have interval total colorings. They also showed that there are bipartite graphs that have no interval total coloring. The smallest known bipartite graph with 26 vertices and maximum degree 18 that is not interval total colorable was obtained by Shashikyan [19].

One of the less-investigated problems related to interval total colorings is a problem of determining the exact values of the minimum and the maximum span in interval total colorings of graphs. The exact values of these parameters are known only for paths, cycles, trees [13, 14], wheels [10], complete and complete balanced bipartite graphs [5, 6, 18]. In some papers [8, 11, 12, 17] lower and upper bounds are found for the minimum and the maximum span in interval total colorings of certain graphs.

In this paper we prove that all complete balanced multipartite graphs are interval total colorable and we derive some bounds for the minimum and the maximum span in interval total colorings of these graphs. Next, we investigate interval total colorings of hypercubes Q_n and we show that $W_\tau(Q_n) = \frac{(n+1)(n+2)}{2}$ for any $n \in \mathbf{N}$.

2.   Interval Total Colorings of Complete Multipartite Graphs

We first consider interval total colorings of complete bipartite graphs. It is known that $K_{m,n} \in \mathcal{T}$ and $w_\tau(K_{m,n}) \leq m+n+2-\gcd(m,n)$ for any $m,n \in \mathbf{N}$, but in general the exact value of $w_\tau(K_{m,n})$ is unknown. Our first result generalizes the point (2) of Theorem 5.

Theorem 8: *For any* $n,l \in \mathbf{N}$, *we have* $w_\tau(K_{n,n \cdot l}) = \chi''(K_{n,n \cdot l})$.

Proof: First of all let us consider the case $l = 1$. By Theorem 5, we obtain $w_\tau(K_{n,n}) = \chi''(K_{n,n}) = n+2$ for any $n \in \mathbf{N}$.

Now we assume that $l \geq 2$.

Let $V(K_{n,n \cdot l}) = \{u_1, \ldots, u_n, v_1, \ldots, v_{n \cdot l}\}$ and $E(K_{n,n \cdot l}) = \{u_i v_j | 1 \leq i \leq n, 1 \leq j \leq n \cdot l\}$. Also, let $G = K_{n,n \cdot l}[\{u_1, \ldots, u_n, v_1, \ldots, v_n\}]$. Clearly, G is isomorphic to the graph $K_{n,n}$.

Now we define an edge-coloring α of G as follows: for $1 \leq i \leq n$ and $1 \leq j \leq n$, let

$$\alpha(u_i v_j) = \begin{cases} i+j-1 \pmod{n}, & \text{if } i+j \neq n+1, \\ n, & \text{if } i+j = n+1. \end{cases}$$

It is easy to see that α is a proper edge-coloring of G and $S(u_i, \alpha) = S(v_i, \alpha) = [1,n]$ for $1 \leq i \leq n$.

Next we construct an interval total $(n \cdot l + 1)$-coloring of $K_{n,n \cdot l}$. Before we give the explicit definition of the coloring, we need two auxiliary functions. For $i \in \mathbf{N}$, we define a function $f_1(i)$ as follows: $f_1(i) = 1 + (i-1) \pmod{n}$. For $j \in \mathbf{N}$, we define a function $f_2(j)$ as follows: $f_2(j) = \left\lfloor \frac{j-1}{n} \right\rfloor$.

Now we are able to define a total coloring β of $K_{n,n \cdot l}$.

For $1 \leq i \leq n$, let

$$\beta(u_i) = n \cdot l + 1.$$

For $1 \leq j \leq n \cdot l$, let

$$\beta(v_j) = \begin{cases} n+1, & \text{if } 1 \leq j \leq n, \\ n \cdot f_2(j), & \text{if } n+1 \leq j \leq n \cdot l. \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq n \cdot l$, let

$$\beta(u_i v_j) = \alpha\left(u_{f_1(i)} v_{f_1(j)}\right) + n \cdot f_2(j).$$

Let us prove that β is an interval total $(n \cdot l + 1)$-coloring of $K_{n,n \cdot l}$.

By the definition of β and taking into account that $S(u_i, \alpha) = S(v_i, \alpha) = [1,n]$ for $1 \leq i \leq n$, we have

$$S[u_i, \beta] = S(u_i, \beta) \cup \{\beta(u_i)\} = \left(\bigcup_{k=1}^{l} S\left(u_{f_1(i)}, \alpha\right) \oplus n(k-1)\right) \cup \{\beta(u_i)\} =$$
$$= [1, n \cdot l] \cup \{n \cdot l + 1\} = [1, n \cdot l + 1] \text{ for } 1 \leq i \leq n,$$
$$S[v_j, \beta] = S(v_j, \alpha) \cup \{\beta(v_j)\} = [1,n] \cup \{n+1\} = [1, n+1] \text{ for } 1 \leq j \leq n, \text{ and}$$
$$S[v_j, \beta] = S(v_j, \beta) \cup \{\beta(v_j)\} = \left(S\left(v_{f_1(j)}, \alpha\right) \oplus n \cdot f_2(j)\right) \cup \{\beta(v_j)\} =$$
$$= [1 + n \cdot f_2(j), n + n \cdot f_2(j)] \cup \{n \cdot f_2(j)\} = [n \cdot f_2(j), n + n \cdot f_2(j)] \text{ for } n+1 \leq j \leq n \cdot l.$$

This shows that β is an interval total $(n \cdot l + 1)$-coloring of $K_{n,n \cdot l}$. Thus, $w_\tau(K_{n,n \cdot l}) \le n \cdot l + 1$. On the other hand, by Theorem 2, $w_\tau(K_{n,n \cdot l}) \ge \chi''(K_{n,n \cdot l}) = n \cdot l + 1$ for $l \ge 2$ and hence, $w_\tau(K_{n,n \cdot l}) = \chi''(K_{n,n \cdot l})$. ∎

Next, we show that the difference between $w_\tau(K_{m,n})$ and $\chi''(K_{m,n})$ for some m and n can be arbitrarily large.

Theorem 9: *For any $l \in \mathbf{N}$, there exists a graph G such that $G \in \mathcal{T}$ and $w_\tau(G) - \chi''(G) \ge l$.*

Proof: Let $n = l + 3$. Clearly, $n \ge 4$. Consider the complete bipartite graph $K_{n+l,2n}$ with bipartition (X, Y), where $|X| = n + l$ and $|Y| = 2n$. By Theorem 2, we have $\chi''(K_{n+l,2n}) = 2n + 1$.

By Theorem 4, we have $K_{n+l,2n} \in \mathcal{T}$. We now show that $w_\tau(K_{n+l,2n}) \ge 2n + 1 + l$.

Suppose, to the contrary, that $K_{n+l,2n}$ has an interval total t-coloring α, where $2n + 1 \le t \le 2n + l$.

Let us consider $S[v, \alpha]$ for any $v \in V(K_{n+l,2n})$. It is easy to see that $[t - n - l, n + l + 1] \subseteq S[v, \alpha]$ for any $v \in V(K_{n+l,2n})$. Since $t \le 2n + l$, we have $[n, n + l + 1] \subseteq S[v, \alpha]$ for any $v \in V(K_{n+l,2n})$. This implies that for each $c \in [n, n + l + 1]$, there are $n - l$ vertices in Y colored by c. On the other hand, since $|Y| = 2n$, we have $2n \ge (l + 2)(n - l)$, which contradicts the equality $n = l + 3$.

This shows that $w_\tau(K_{n+l,2n}) \ge 2n + 1 + l$. We take $G = K_{n+l,2n}$. Hence, $w_\tau(G) - \chi''(G) \ge l$. ∎

Now we consider interval total colorings of complete r-partite graphs with n vertices in each part.

Theorem 10: *If $r = 2$ or r is even and n is odd, then $K_{n,\ldots,n} \in \mathcal{T}$ and*

$$w_\tau(K_{n,\ldots,n}) \le \left(\tfrac{3}{2}r - 2\right)n + 2.$$

Proof: First let us note that the theorem is true for the case $r = 2$, since $w_\tau(K_{n,n}) = \chi''(K_{n,n}) = n + 2$ for any $n \in \mathbf{N}$, by Theorem 5.

Now we assume that r is even and n is odd. Clearly, for the proof of the theorem, it suffices to prove that $K_{n,\ldots,n}$ has an interval total $\left(\left(\tfrac{3}{2}r - 2\right)n + 2\right)$-coloring.

Let $V(K_{n,\ldots,n}) = \left\{v_j^{(i)} \mid 1 \le i \le r, 1 \le j \le n\right\}$ and $E(K_{n,\ldots,n}) = \left\{v_p^{(i)} v_q^{(j)} \mid 1 \le i < j \le r, 1 \le p \le n, 1 \le q \le n\right\}$.

Define a total coloring α of $K_{n,\ldots,n}$. First we color the vertices of the graph as follows:

$$\alpha\left(v_j^{(1)}\right) = 1 \text{ for } 1 \le j \le n \text{ and } \alpha\left(v_j^{(2)}\right) = (r - 1)n + 2 \text{ for } 1 \le j \le n,$$

$$\alpha\left(v_j^{(i+1)}\right) = (i - 1)n + 1 \text{ for } 2 \le i \le \tfrac{r}{2} - 1 \text{ and } 1 \le j \le n,$$

$$\alpha\left(v_j^{(\frac{r}{2}+i-1)}\right) = (r + i - 2)n + 2 \text{ for } 2 \le i \le \tfrac{r}{2} - 1 \text{ and } 1 \le j \le n,$$

$$\alpha\left(v_j^{(r-1)}\right) = \left(\tfrac{r}{2} - 1\right)n + 1 \text{ for } 1 \le j \le n \text{ and } \alpha\left(v_j^{(r)}\right) = \left(\tfrac{3}{2}r - 2\right)n + 2 \text{ for } 1 \le j \le n.$$

Next we color the edges of the graph. For each edge $v_p^{(i)} v_q^{(j)} \in E(K_{n,\ldots,n})$ with $1 \le i < j \le r$ and $p = 1, \ldots, n$, $q = 1, \ldots, n$, define a color $\alpha\left(v_p^{(i)} v_q^{(j)}\right)$ as follows:

for $i = 1, \ldots, \left\lfloor \frac{r}{4} \right\rfloor$, $j = 2, \ldots, \frac{r}{2}$, $i + j \leq \frac{r}{2} + 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - 3) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = 2, \ldots, \frac{r}{2} - 1$, $j = \left\lfloor \frac{r}{4} \right\rfloor + 2, \ldots, \frac{r}{2}$, $i + j \geq \frac{r}{2} + 2$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(i + j + \frac{r}{2} - 4\right) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = 3, \ldots, \frac{r}{2}$, $j = \frac{r}{2} + 1, \ldots, r - 2$, $j - i \leq \frac{r}{2} - 2$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(\frac{r}{2} + j - i - 1\right) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = 1, \ldots, \frac{r}{2}$, $j = \frac{r}{2} + 1, \ldots, r$, $j - i \geq \frac{r}{2}$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (j - i - 1) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = 2, \ldots, 1 + \left\lfloor \frac{r-2}{4} \right\rfloor$, $j = \frac{r}{2} + 1, \ldots, \frac{r}{2} + \left\lfloor \frac{r-2}{4} \right\rfloor$, $j - i = \frac{r}{2} - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (2i - 3) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = \left\lfloor \frac{r-2}{4} \right\rfloor + 2, \ldots, \frac{r}{2}$, $j = \frac{r}{2} + 1 + \left\lfloor \frac{r-2}{4} \right\rfloor, \ldots, r - 1$, $j - i = \frac{r}{2} - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - 3) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = \frac{r}{2} + 1, \ldots, \frac{r}{2} + \left\lfloor \frac{r}{4} \right\rfloor - 1$, $j = \frac{r}{2} + 2, \ldots, r - 2$, $i + j \leq \frac{3}{2} r - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - r - 1) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1; \end{cases}$$

for $i = \frac{r}{2} + 1, \ldots, r - 1$, $j = \frac{r}{2} + \left\lfloor \frac{r}{4} \right\rfloor + 1, \ldots, r$, $i + j \geq \frac{3}{2} r$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(i + j - \frac{r}{2} - 2\right) n + \begin{cases} 1 + (p + q - 1) \pmod{n}, & \text{if } p + q \neq n + 1, \\ n + 1, & \text{if } p + q = n + 1. \end{cases}$$

Let us prove that α is an interval total $\left(\left(\frac{3}{2} r - 2\right) n + 2\right)$-coloring of $K_{n,\ldots,n}$.

First we show that for each $c \in \left[1, \left(\frac{3}{2} r - 2\right) n + 2\right]$, there is $ve \in VE(K_{n,\ldots,n})$ with $\alpha(ve) = c$.

Consider the vertices $v_1^{(1)}$ and $v_1^{(r)}$. By the definition of α, we have

$$S\left[v_1^{(1)}, \alpha\right] = S\left(v_1^{(1)}, \alpha\right) \cup \left\{\alpha\left(v_1^{(1)}\right)\right\} = \left(\bigcup_{l=1}^{r-1} ([2, n+1] \oplus (l-1)n)\right) \cup \{1\} =$$
$$= [2, (r-1)n + 1] \cup \{1\} = [1, (r-1)n + 1] \text{ and}$$

$$S\left[v_1^{(r)}, \alpha\right] = S\left(v_1^{(r)}, \alpha\right) \cup \left\{\alpha\left(v_1^{(r)}\right)\right\} = \left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2} ([2, n+1] \oplus (l-1)n)\right) \cup \left\{\left(\frac{3}{2} r - 2\right) n + 2\right\} =$$
$$= \left[\left(\frac{r}{2} - 1\right) n + 2, \left(\frac{3}{2} r - 2\right) n + 1\right] \cup \left\{\left(\frac{3}{2} r - 2\right) n + 2\right\} = \left[\left(\frac{r}{2} - 1\right) n + 2, \left(\frac{3}{2} r - 2\right) n + 2\right].$$

It is straightforward to check that $S\left[v_1^{(1)}, \alpha\right] \cup S\left[v_1^{(r)}, \alpha\right] = \left[1, \left(\frac{3}{2}r - 2\right)n + 2\right]$, so for each $c \in \left[1, \left(\frac{3}{2}r - 2\right)n + 2\right]$, there is $ve \in VE(K_{n,\dots,n})$ with $\alpha(ve) = c$.

Next we show that the edges incident to each vertex of $K_{n,\dots,n}$ together with this vertex are colored by $(r-1)n + 1$ consecutive colors.

Let $v_j^{(i)} \in V(K_{n,\dots,n})$, where $1 \leq i \leq r$, $1 \leq j \leq n$.

Case 1. $1 \leq i \leq 2$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(1)}, \alpha\right] = S\left(v_j^{(1)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(1)}\right)\right\} = \left(\bigcup_{l=1}^{r-1}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \{1\} =$$
$$= [2, (r-1)n + 1] \cup \{1\} = [1, (r-1)n + 1] \text{ and}$$
$$S\left[v_j^{(2)}, \alpha\right] = S\left(v_j^{(2)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(2)}\right)\right\} = \left(\bigcup_{l=1}^{r-1}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \{(r-1)n + 2\} =$$
$$= [2, (r-1)n + 1] \cup \{(r-1)n + 2\} = [2, (r-1)n + 2].$$

Case 2. $3 \leq i \leq \frac{r}{2}$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(i)}, \alpha\right] = S\left(v_j^{(i)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(i)}\right)\right\} = \left(\bigcup_{l=i-1}^{r-3+i}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \{(i-2)n + 1\} =$$
$$= [(i-2)n + 2, (r-3+i)n + 1] \cup \{(i-2)n + 1\} = [(i-2)n + 1, (r-3+i)n + 1].$$

Case 3. $\frac{r}{2} + 1 \leq i \leq r - 2$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(i)}, \alpha\right] = S\left(v_j^{(i)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(i)}\right)\right\} =$$
$$\left(\bigcup_{l=i-\frac{r}{2}+1}^{\frac{r}{2}-1+i}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \left\{\left(\frac{r}{2} + i - 1\right)n + 2\right\} =$$
$$\left[\left(i - \frac{r}{2}\right)n + 2, \left(\frac{r}{2} + i - 1\right)n + 1\right] \cup \left\{\left(\frac{r}{2} + i - 1\right)n + 2\right\} = \left[\left(i - \frac{r}{2}\right)n + 2, \left(\frac{r}{2} + i - 1\right)n + 2\right].$$

Case 4. $r - 1 \leq i \leq r$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(r-1)}, \alpha\right] = S\left(v_j^{(r-1)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(r-1)}\right)\right\} =$$
$$\left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \left\{\left(\frac{r}{2} - 1\right)n + 1\right\} =$$
$$\left[\left(\frac{r}{2} - 1\right)n + 2, \left(\frac{3}{2}r - 2\right)n + 1\right] \cup \left\{\left(\frac{r}{2} - 1\right)n + 1\right\} = \left[\left(\frac{r}{2} - 1\right)n + 1, \left(\frac{3}{2}r - 2\right)n + 1\right] \text{ and}$$
$$S\left[v_j^{(r)}, \alpha\right] = S\left(v_j^{(r)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(r)}\right)\right\} = \left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2}\left([2, n+1] \oplus (l-1)n\right)\right) \cup \left\{\left(\frac{3}{2}r - 2\right)n + 2\right\} =$$
$$\left[\left(\frac{r}{2} - 1\right)n + 2, \left(\frac{3}{2}r - 2\right)n + 1\right] \cup \left\{\left(\frac{3}{2}r - 2\right)n + 2\right\} = \left[\left(\frac{r}{2} - 1\right)n + 2, \left(\frac{3}{2}r - 2\right)n + 2\right].$$

This shows that α is an interval total $\left(\left(\frac{3}{2}r - 2\right)n + 2\right)$-coloring of $K_{n,\dots,n}$; thus, $K_{n,\dots,n} \in \mathcal{T}$ and $w_\tau(K_{n,\dots,n}) \leq \left(\frac{3}{2}r - 2\right)n + 2$. ∎

Note that the upper bound in Theorem 10 is sharp when $r = 2$ or $n = 1$. Also, by Theorem 10, we have that if $r = 2$ or r is even and n is odd, then $K_{n,\dots,n} \in \mathcal{T}$; otherwise, by Theorem 3 and taking into account that $K_{n,\dots,n}$ is an $(r-1)n$-regular graph, we have $K_{n,\dots,n} \in \mathcal{T}$ and $w_\tau(K_{n,\dots,n}) = \chi''(K_{n,\dots,n}) = (r-1)n + 1$. Our next result gives a lower bound for $W_\tau(K_{n,\dots,n})$ when $n \cdot r$ is even.

Theorem 11: If $r \geq 2, n \in \mathbf{N}$ and $n \cdot r$ is even, then $W_\tau(K_{n,\dots,n}) \geq \left(\frac{3}{2}r - 1\right)n + 1$.

Proof: We consider two cases.

Case 1: r is even.

Let $V(K_{n,\ldots,n}) = \left\{ v_j^{(i)} \mid 1 \le i \le r, 1 \le j \le n \right\}$ and

$E(K_{n,\ldots,n}) = \left\{ v_p^{(i)} v_q^{(j)} \mid 1 \le i < j \le r, 1 \le p \le n, 1 \le q \le n \right\}.$

Define a total coloring α of $K_{n,\ldots,n}$. First we color the vertices of the graph as follows:

$$\alpha\left(v_j^{(1)}\right) = j \text{ for } 1 \le j \le n \text{ and } \alpha\left(v_j^{(2)}\right) = (r-1)n + 1 + j \text{ for } 1 \le j \le n,$$

$$\alpha\left(v_j^{(i+1)}\right) = (i-1)n + j \text{ for } 2 \le i \le \tfrac{r}{2} - 1 \text{ and } 1 \le j \le n,$$

$$\alpha\left(v_j^{(\frac{r}{2}+i-1)}\right) = (r+i-2)n + 1 + j \text{ for } 2 \le i \le \tfrac{r}{2} - 1 \text{ and } 1 \le j \le n,$$

$$\alpha\left(v_j^{(r-1)}\right) = \left(\tfrac{r}{2} - 1\right)n + j \text{ for } 1 \le j \le n \text{ and } \alpha\left(v_j^{(r)}\right) = \left(\tfrac{3}{2}r - 2\right)n + 1 + j \text{ for } 1 \le j \le n.$$

Next we color the edges of the graph. For each edge $v_p^{(i)} v_q^{(j)} \in E(K_{n,\ldots,n})$ with $1 \le i < j \le r$ and $p = 1, \ldots, n$, $q = 1, \ldots, n$, define a color $\alpha\left(v_p^{(i)} v_q^{(j)}\right)$ as follows:

for $i = 1, \ldots, \left\lfloor \tfrac{r}{4} \right\rfloor$, $j = 2, \ldots, \tfrac{r}{2}$, $i + j \le \tfrac{r}{2} + 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - 3)n + p + q;$$

for $i = 2, \ldots, \tfrac{r}{2} - 1$, $j = \left\lfloor \tfrac{r}{4} \right\rfloor + 2, \ldots, \tfrac{r}{2}$, $i + j \ge \tfrac{r}{2} + 2$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(i + j + \tfrac{r}{2} - 4\right)n + p + q;$$

for $i = 3, \ldots, \tfrac{r}{2}$, $j = \tfrac{r}{2} + 1, \ldots, r - 2$, $j - i \le \tfrac{r}{2} - 2$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(\tfrac{r}{2} + j - i - 1\right)n + p + q;$$

for $i = 1, \ldots, \tfrac{r}{2}$, $j = \tfrac{r}{2} + 1, \ldots, r$, $j - i \ge \tfrac{r}{2}$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (j - i - 1)n + p + q;$$

for $i = 2, \ldots, 1 + \left\lfloor \tfrac{r-2}{4} \right\rfloor$, $j = \tfrac{r}{2} + 1, \ldots, \tfrac{r}{2} + \left\lfloor \tfrac{r-2}{4} \right\rfloor$, $j - i = \tfrac{r}{2} - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (2i - 3)n + p + q;$$

for $i = \left\lfloor \tfrac{r-2}{4} \right\rfloor + 2, \ldots, \tfrac{r}{2}$, $j = \tfrac{r}{2} + 1 + \left\lfloor \tfrac{r-2}{4} \right\rfloor, \ldots, r - 1$, $j - i = \tfrac{r}{2} - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - 3)n + p + q;$$

for $i = \tfrac{r}{2} + 1, \ldots, \tfrac{r}{2} + \left\lfloor \tfrac{r}{4} \right\rfloor - 1$, $j = \tfrac{r}{2} + 2, \ldots, r - 2$, $i + j \le \tfrac{3}{2}r - 1$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = (i + j - r - 1)n + p + q;$$

for $i = \tfrac{r}{2} + 1, \ldots, r - 1$, $j = \tfrac{r}{2} + \left\lfloor \tfrac{r}{4} \right\rfloor + 1, \ldots, r$, $i + j \ge \tfrac{3}{2}r$, let

$$\alpha\left(v_p^{(i)} v_q^{(j)}\right) = \left(i + j - \tfrac{r}{2} - 2\right) n + p + q.$$

Let us prove that α is an interval total $\left(\left(\tfrac{3}{2}r - 1\right) n + 1\right)$-coloring of $K_{n,\ldots,n}$.

First we show that for each $c \in \left[1, \left(\tfrac{3}{2}r - 1\right) n + 1\right]$, there is $ve \in VE(K_{n,\ldots,n})$ with $\alpha(ve) = c$.

Consider the vertices $v_1^{(1)}, \ldots, v_n^{(1)}$ and $v_1^{(r)}, \ldots, v_n^{(r)}$. By the definition of α, for $1 \leq j \leq n$, we have

$$S\left[v_j^{(1)}, \alpha\right] = S\left(v_j^{(1)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(1)}\right)\right\} = \left(\bigcup_{l=1}^{r-1} ([j+1, j+n] \oplus (l-1)n)\right) \cup \{j\} =$$
$$= [j+1, (r-1)n + j] \cup \{j\} = [j, (r-1)n + j] \text{ and } S\left[v_j^{(r)}, \alpha\right] = S\left(v_j^{(r)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(r)}\right)\right\} =$$
$$= \left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2} ([j+1, j+n] \oplus (l-1)n)\right) \cup \left\{\left(\tfrac{3}{2}r - 2\right) n + 1 + j\right\} =$$
$$= \left[\left(\tfrac{r}{2} - 1\right) n + 1 + j, \left(\tfrac{3}{2}r - 2\right) n + j\right] \cup \left\{\left(\tfrac{3}{2}r - 2\right) n + 1 + j\right\} =$$
$$= \left[\left(\tfrac{r}{2} - 1\right) n + 1 + j, \left(\tfrac{3}{2}r - 2\right) n + 1 + j\right].$$

Let $\overline{C} = \bigcup_{j=1}^n S\left[v_j^{(1)}, \alpha\right]$ and $\overline{\overline{C}} = \bigcup_{j=1}^n S\left[v_j^{(r)}, \alpha\right]$. It is straightforward to check that $\overline{C} \cup \overline{\overline{C}} = \left[1, \left(\tfrac{3}{2}r - 1\right) n + 1\right]$, so for each $c \in \left[1, \left(\tfrac{3}{2}k - 1\right) n + 1\right]$, there is $ve \in VE(K_{n,\ldots,n})$ with $\alpha(ve) = c$.

Next we show that the edges incident to each vertex of $K_{n,\ldots,n}$ together with this vertex are colored by $(r-1)n + 1$ consecutive colors.

Let $v_j^{(i)} \in V(K_{n,\ldots,n})$, where $1 \leq i \leq r$, $1 \leq j \leq n$.

Subcase 1.1. $1 \leq i \leq 2$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(1)}, \alpha\right] = S\left(v_j^{(1)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(1)}\right)\right\} = \left(\bigcup_{l=1}^{r-1} ([j+1, j+n] \oplus (l-1)n)\right) \cup \{j\} =$$
$$= [j+1, (r-1)n + j] \cup \{j\} = [j, (r-1)n + j] \text{ and}$$
$$S\left[v_j^{(2)}, \alpha\right] = S\left(v_j^{(2)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(2)}\right)\right\} = \left(\bigcup_{l=1}^{r-1} ([j+1, j+n] \oplus (l-1)n)\right) \cup \{(r-1)n + 1 + j\} =$$
$$= [j+1, (r-1)n + 1 + j].$$

Subcase 1.2. $3 \leq i \leq \tfrac{r}{2}$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(i)}, \alpha\right] = S\left(v_j^{(i)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(i)}\right)\right\} = \left(\bigcup_{l=i-1}^{r-3+i} ([j+1, j+n] \oplus (l-1)n)\right) \cup \{(i-2)n + j\} =$$
$$= [(i-2)n + 1 + j, (r-3+i)n + j] \cup \{(i-2)n + j\} = [(i-2)n + j, (r-3+i)n + j].$$

Subcase 1.3. $\tfrac{r}{2} + 1 \leq i \leq r - 2$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(i)}, \alpha\right] = S\left(v_j^{(i)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(i)}\right)\right\} =$$
$$= \left(\bigcup_{l=i-\frac{r}{2}+1}^{\frac{r}{2}-1+i} ([j+1, j+n] \oplus (l-1)n)\right) \cup \left\{\left(\tfrac{r}{2} + i - 1\right) n + 1 + j\right\} =$$
$$= \left[\left(i - \tfrac{r}{2}\right) n + 1 + j, \left(\tfrac{r}{2} + i - 1\right) n + j\right] \cup \left\{\left(\tfrac{r}{2} + i - 1\right) n + 1 + j\right\} =$$
$$\left[\left(i - \tfrac{r}{2}\right) n + 1 + j, \left(\tfrac{r}{2} + i - 1\right) n + 1 + j\right].$$

Subcase 1.4. $r - 1 \leq i \leq r$, $1 \leq j \leq n$.

By the definition of α, we have

$$S\left[v_j^{(r-1)}, \alpha\right] = S\left(v_j^{(r-1)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(r-1)}\right)\right\} =$$
$$\left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2} ([j+1, j+n] \oplus (l-1)n)\right) \cup \left\{\left(\frac{r}{2}-1\right)n + j\right\} =$$
$$\left[\left(\frac{r}{2}-1\right)n + 1 + j, \left(\frac{3}{2}r - 2\right)n + j\right] \cup \left\{\left(\frac{r}{2}-1\right)n + j\right\} = \left[\left(\frac{r}{2}-1\right)n + j, \left(\frac{3}{2}r - 2\right)n + j\right] \text{ and}$$
$$S\left[v_j^{(r)}, \alpha\right] = S\left(v_j^{(r)}, \alpha\right) \cup \left\{\alpha\left(v_j^{(r)}\right)\right\} = \left(\bigcup_{l=\frac{r}{2}}^{\frac{3}{2}r-2} ([j+1, j+n] \oplus (l-1)n)\right) \cup$$
$$\left\{\left(\frac{3}{2}r - 2\right)n + 1 + j\right\} = \left[\left(\frac{r}{2}-1\right)n + 1 + j, \left(\frac{3}{2}r - 2\right)n + j\right] \cup \left\{\left(\frac{3}{2}r - 2\right)n + 1 + j\right\} =$$
$$\left[\left(\frac{r}{2}-1\right)n + 1 + j, \left(\frac{3}{2}r - 2\right)n + 1 + j\right].$$

This shows that α is an interval total $\left(\left(\frac{3}{2}r - 1\right)n + 1\right)$-coloring of $K_{n,\dots,n}$; thus, $W_\tau(K_{n,\dots,n}) \geq \left(\frac{3}{2}r - 1\right)n + 1$ for even $r \geq 2$.

Case 2: n is even.

Let $n = 2m$, $m \in \mathbf{N}$. Let $S_i = \left\{u_1^{(i)}, \dots, u_m^{(i)}, u_1^{(r+i)}, \dots, u_m^{(r+i)}\right\}$ $(1 \leq i \leq r)$ be the r independent sets of vertices of $K_{n,\dots,n}$. For $i = 1, \dots, 2r$, define the set U_i as follows: $U_i = \left\{u_1^{(i)}, \dots, u_m^{(i)}\right\}$. Clearly, $V(K_{n,\dots,n}) = \bigcup_{i=1}^{2r} U_i$. For $1 \leq i < j \leq 2r$, define (U_i, U_j) as the set of all edges between U_i and U_j. It is easy to see that for $1 \leq i < j \leq 2r$, $|(U_i, U_j)| = m^2$ except for $|(U_i, U_{r+i})| = 0$ whenever $i = 1, \dots, r$. If we consider the sets U_i as the vertices and the sets (U_i, U_j) as the edges, then we obtain that $K_{n,\dots,n}$ is isomorphic to the graph $K_{2r} - F$, where F is a perfect matching of K_{2r}. Now we define a total coloring β of the graph $K_{n,\dots,n}$. First we color the vertices of the graph as follows:

$$\beta\left(u_j^{(1)}\right) = j \text{ for } 1 \leq j \leq m \text{ and } \beta\left(u_j^{(2)}\right) = (2r-2)m + 1 + j \text{ for } 1 \leq j \leq m,$$
$$\beta\left(u_j^{(i+1)}\right) = (i-1)m + j \text{ for } 2 \leq i \leq r-1 \text{ and } 1 \leq j \leq m,$$
$$\beta\left(u_j^{(r+i-1)}\right) = (2r+i-3)m + 1 + j \text{ for } 2 \leq i \leq r-1 \text{ and } 1 \leq j \leq m,$$
$$\beta\left(u_j^{(2r-1)}\right) = (r-1)m + j \text{ for } 1 \leq j \leq m \text{ and } \beta\left(u_j^{(2r)}\right) = (3r-3)m + 1 + j \text{ for } 1 \leq j \leq m.$$

Next we color the edges of the graph. For each edge $u_p^{(i)}u_q^{(j)} \in E(K_{n,\dots,n})$ with $1 \leq i < j \leq 2r$ and $p = 1, \dots, m$, $q = 1, \dots, m$, define a color $\beta\left(u_p^{(i)}u_q^{(j)}\right)$ as follows:

for $i = 1, \dots, \left\lfloor\frac{r}{2}\right\rfloor$, $j = 2, \dots, r$, $i + j \leq r + 1$, let

$$\beta\left(u_p^{(i)}u_q^{(j)}\right) = (i+j-3)m + p + q;$$

for $i = 2, \dots, r-1$, $j = \left\lfloor\frac{r}{2}\right\rfloor + 2, \dots, r$, $i + j \geq r + 2$, let

$$\beta\left(u_p^{(i)}u_q^{(j)}\right) = (i+j+r-5)m + p + q;$$

for $i = 3, \dots, r$, $j = r+1, \dots, 2r-2$, $j - i \leq r - 2$, let

$$\beta\left(u_p^{(i)}u_q^{(j)}\right) = (r+j-i-2)m + p + q;$$

for $i = 1, \dots, r-1$, $j = r+2, \dots, 2r$, $j - i \geq r + 1$, let

$$\beta\left(u_p^{(i)}u_q^{(j)}\right) = (j-i-2)m + p + q;$$

for $i = 2, \ldots, 1 + \left\lfloor \frac{r-1}{2} \right\rfloor$, $j = r + 1, \ldots, r + \left\lfloor \frac{r-1}{2} \right\rfloor$, $j - i = r - 1$, let

$$\beta \left(u_p^{(i)} u_q^{(j)} \right) = (2i - 3) m + p + q;$$

for $i = \left\lfloor \frac{r-1}{2} \right\rfloor + 2, \ldots, r$, $j = r + 1 + \left\lfloor \frac{r-1}{2} \right\rfloor, \ldots, 2r - 1$, $j - i = r - 1$, let

$$\beta \left(u_p^{(i)} u_q^{(j)} \right) = (i + j - 4) m + p + q;$$

for $i = r + 1, \ldots, r + \left\lfloor \frac{r}{2} \right\rfloor - 1$, $j = r + 2, \ldots, 2r - 2$, $i + j \leq 3r - 1$, let

$$\beta \left(u_p^{(i)} u_q^{(j)} \right) = (i + j - 2r - 1) m + p + q;$$

for $i = r + 1, \ldots, 2r - 1$, $j = r + \left\lfloor \frac{r}{2} \right\rfloor + 1, \ldots, 2r$, $i + j \geq 3r$, let

$$\beta \left(u_p^{(i)} u_q^{(j)} \right) = (i + j - r - 3) m + p + q.$$

Let us prove that β is an interval total $\left(\left(\frac{3}{2}r - 1 \right) n + 1 \right)$-coloring of the graph $K_{n,\ldots,n}$.

First we show that for each $c \in \left[1, \left(\frac{3}{2}r - 1 \right) n + 1 \right]$, there is $ve \in VE(K_{n,\ldots,n})$ with $\beta(ve) = c$.

Consider the vertices $u_1^{(1)}, \ldots, u_m^{(1)}$ and $u_1^{(2r)}, \ldots, u_m^{(2r)}$. By the definition of β, for $1 \leq j \leq m$, we have

$$S \left[u_j^{(1)}, \beta \right] = S \left(u_j^{(1)}, \beta \right) \cup \left\{ \beta \left(u_j^{(1)} \right) \right\} = \left(\bigcup_{l=1}^{2r-2} ([j+1, j+m] \oplus (l-1)m) \right) \cup \{j\} =$$
$$[j+1, (2r-2)m + j] \cup \{j\} = [j, (2r-2)m + j] \text{ and}$$
$$S \left[u_j^{(2r)}, \beta \right] = S \left(u_j^{(2r)}, \beta \right) \cup \left\{ \beta \left(u_j^{(2r)} \right) \right\} = \left(\bigcup_{l=r}^{3r-3} ([j+1, j+m] \oplus (l-1)m) \right) \cup \{(3r-3)m +$$
$$1 + j\} = [(r-1)m + 1 + j, (3r-3)m + j] \cup \{(3r-3)m + 1 + j\} = [(r-1)m + 1 + j, (3r-3)m + 1 + j].$$

Let $\tilde{C} = \bigcup_{j=1}^{m} S \left[u_j^{(1)}, \beta \right]$ and $\tilde{\tilde{C}} = \bigcup_{j=1}^{m} S \left[u_j^{(2r)}, \beta \right]$. It is straightforward to check that $\tilde{C} \cup \tilde{\tilde{C}} = \left[1, \left(\frac{3}{2}r - 1 \right) n + 1 \right]$, so for each $c \in \left[1, \left(\frac{3}{2}r - 1 \right) n + 1 \right]$, there is $ve \in VE(K_{n,\ldots,n})$ with $\beta(ve) = c$.

Next we show that the edges incident to each vertex of $K_{n,\ldots,n}$ together with this vertex are colored by $(r-1)n + 1$ consecutive colors.

Let $v_j^{(i)} \in V(K_{n,\ldots,n})$, where $1 \leq i \leq 2r$, $1 \leq j \leq m$.

Subcase 2.1. $1 \leq i \leq 2$, $1 \leq j \leq m$.

By the definition of β, we have

$$S \left[v_j^{(1)}, \beta \right] = S \left(u_j^{(1)}, \beta \right) \cup \left\{ \beta \left(u_j^{(1)} \right) \right\} = \left(\bigcup_{l=1}^{2r-2} ([j+1, j+m] \oplus (l-1)m) \right) \cup \{j\} =$$
$$[j+1, (2r-2)m + j] \cup \{j\} = [j, (2r-2)m + j] \text{ and}$$
$$S \left[u_j^{(2)}, \beta \right] = S \left(u_j^{(2)}, \beta \right) \cup \left\{ \beta \left(u_j^{(2)} \right) \right\} = \left(\bigcup_{l=1}^{2r-2} ([j+1, j+m] \oplus (l-1)m) \right) \cup \{(2r-2)m +$$
$$1 + j\} = [j+1, (2r-2)m + j] \cup \{(2r-2)m + 1 + j\} = [j+1, (2r-2)m + 1 + j].$$

Subcase 2.2. $3 \leq i \leq r$, $1 \leq j \leq m$.

By the definition of β, we have

$$S\left[u_j^{(i)}, \beta\right] = S\left(u_j^{(i)}, \beta\right) \cup \left\{\beta\left(u_j^{(i)}\right)\right\} = \left(\bigcup_{l=i-1}^{2r-4+i}\left([j+1, j+m] \oplus (l-1)m\right)\right) \cup \{(i-2)m+j\} = [(i-2)m+1+j, (2r-4+i)m+j] \cup \{(i-2)m+j\} = [(i-2)m+j, (2r-4+i)m+j].$$

Subcase 2.3. $r+1 \le i \le 2r-2$, $1 \le j \le m$.

By the definition of β, we have

$$S\left[u_j^{(i)}, \beta\right] = S\left(u_j^{(i)}, \beta\right) \cup \left\{\beta\left(u_j^{(i)}\right)\right\} = \left(\bigcup_{l=i-r+1}^{r-2+i}\left([j+1, j+m] \oplus (l-1)m\right)\right) \cup \{(r+i-2)m+1+j\} = [(i-r)m+1+j, (r+i-2)m+j] \cup \{(r+i-2)m+1+j\} = [(i-r)m+1+j, (r+i-2)m+1+j].$$

Subcase 2.4. $2r-1 \le i \le 2r$, $1 \le j \le m$.

By the definition of β, we have

$$S\left[u_j^{(2r-1)}, \beta\right] = S\left(u_j^{(2r-1)}, \beta\right) \cup \left\{\beta\left(u_j^{(2r-1)}\right)\right\} = \left(\bigcup_{l=r}^{3r-3}\left([j+1, j+m] \oplus (l-1)m\right)\right) \cup \{(r-1)m+j\} = [(r-1)m+1+j, (3r-3)m+j] \cup \{(r-1)m+j\} = [(r-1)m+j, (3r-3)m+j]$$

and

$$S\left[u_j^{(2r)}, \beta\right] = S\left(u_j^{(2r)}, \beta\right) \cup \left\{\beta\left(u_j^{(2r)}\right)\right\} = \left(\bigcup_{l=r}^{3r-3}\left([j+1, j+m] \oplus (l-1)m\right)\right) \cup \{(3r-3)m+1+j\} = [(r-1)m+1+j, (3r-3)m+j] \cup \{(3r-3)m+1+j\} = [(r-1)m+1+j, (3r-3)m+1+j].$$

This shows that β is an interval total $\left(\left(\frac{3}{2}r-1\right)n+1\right)$-coloring of $K_{n,\dots,n}$; thus, $W_\tau(K_{n,\dots,n}) \ge \left(\frac{3}{2}r-1\right)n+1$ for even $n \ge 2$. ∎

3. Interval Total Colorings of Hypercubes

In [7], the first author investigated interval colorings of hypercubes Q_n. In particular, he proved that $Q_n \in \mathcal{N}$ and $w(Q_n) = n$, $W(Q_n) \ge \frac{n(n+1)}{2}$ for any $n \in \mathbf{N}$. In the same paper he also conjectured that $W(Q_n) = \frac{n(n+1)}{2}$ for any $n \in \mathbf{N}$. In [9], the authors confirmed this conjecture. Here, we prove that $W_\tau(Q_n) = \frac{(n+1)(n+2)}{2}$ for any $n \in \mathbf{N}$.

Theorem 12: If $n \in \mathbf{N}$, then $W_\tau(Q_n) = \frac{(n+1)(n+2)}{2}$.

Proof: First of all let us note that $W_\tau(Q_n) \ge \frac{(n+1)(n+2)}{2}$ for any $n \in \mathbf{N}$, by Theorem 7. For the proof of the theorem, it suffices to show that $W_\tau(Q_n) \le \frac{(n+1)(n+2)}{2}$ for any $n \in \mathbf{N}$. Let φ be an interval total $W_\tau(Q_n)$-coloring of Q_n.

Let $i = 0$ or 1 and $Q_{n+1}^{(i)}$ be a subgraph of the graph Q_{n+1}, induced by the vertices $\{(i, \alpha_2, \alpha_3, \dots, \alpha_{n+1}) \mid (\alpha_2, \alpha_3, \dots, \alpha_{n+1}) \in \{0,1\}^n\}$. Clearly, $Q_{n+1}^{(i)}$ is isomorphic to Q_n for $i \in \{0, 1\}$.

Let us define an edge coloring ψ of the graph Q_{n+1} in the following way:

(1) for $i = 0, 1$ and every edge $(i, \bar\alpha)\left(i, \bar\beta\right) \in E\left(Q_{n+1}^{(i)}\right)$, let

$$\psi\left((i, \bar\alpha)\left(i, \bar\beta\right)\right) = \varphi\left(\bar\alpha\bar\beta\right);$$

(2) for every $\bar\alpha \in \{0,1\}^n$, let

$$\psi\left((0,\bar{\alpha})\,(1,\bar{\alpha})\right)=\varphi\left(\bar{\alpha}\right).$$

It is not difficult to see that ψ is an interval $W_\tau(Q_n)$-coloring of the graph Q_{n+1}. Thus, $W_\tau(Q_n)\leq W(Q_{n+1})=\frac{(n+1)(n+2)}{2}$ for any $n\in\mathbf{N}$. ∎

By Theorems 7 and 12, we have that Q_n has an interval total t-coloring if and only if $w_\tau(Q_n)\leq t\leq W_\tau(Q_n)$.

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Լրիվ բազմակողմանի գրաֆների և հիպերխորանարդների միջակայքային լիակատար ներկումներ

Պ. Պետրոսյան և Ն. Խաչատրյան

Ամփոփում

G գրաֆի լիակատար ներկումը այդ գրաֆի այն ներկումն է, որի դեպքում հարևան գագաթները և կողերը ներկված են տարբեր գույներով և ցանկացած գագաթ և նրան կից կողերը ևս ներկված են տարբեր գույներով: G գրաֆի լիակատար ներկումը $1, \ldots, t$ գույներով կանվանենք միջակայքային լիակատար t–ներկում, եթե բոլոր գույները օգտագործվել են, և յուրաքանչյուր v գագաթին կից կողերը և այդ գագաթը ներկված են $d_G(v) + 1$ հաջորդական գույներով, որտեղ $d_G(v)$-ով նշանակված է v գագաթի աստիճանը G գրաֆում: Այս աշխատանքում ապացուցվում է, որ յուրաքանչյուր կողմում մինևույն քանակի գագաթներ պարունակող լրիվ բազմակողմանի գրաֆներն ունեն միջակայքային լիակատար ներկում: Ավելին, աշխատանքում տրվում են որոշ գնահատականներ այս գրաֆների միջակայքային լիակատար ներկումներում մասնակցող գույների նվազագույն և առավելագույն հնարավոր թվի համար: Աշխատանքում ուսումնասիրվում են նաև Q_n հիպերխորանարդների միջակայքային լիակատար ներկումներ: Մասնավորապես, ապացուցվում է, որ Q_n ($n \geq 3$) հիպերխորանարդ ունի միջակայքային լիակատար t–ներկում այն և միայն այն դեպքում, երբ $n + 1 \leq t \leq \frac{(n+1)(n+2)}{2}$

Интервальные тотальные раскраски полных многодольных графов и гиперкубов

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Аннотация

Тотальной раскраской графа G назовем такую раскраску вершин и ребер графа G, при которой смежные вершины, смежные ребра и вершины, инцидентные ребрам, окрашены в различные цвета. Интервальной тотальной $t-$ раскраской графа G назовем тотальную раскраску графа G в цвета $1, \ldots, t$ при которой все цвета использованы, и ребра, инцидентные каждой вершине v вместе с v, окрашены в $d_G(v) + 1$ последовательных цветов, где $d_G(v)$ - это степень вершины v в графе G. В настоящей работе доказано, что все полные многодольные графы с одинаковым количеством вершин в каждой доле имеют интервальную тотальную раскраску. Кроме того, получены некоторые оценки наименьшего и наибольшего возможного числа цветов в интервальных тотальных раскрасках этих графов. В работе также исследованы интервальные тотальные раскраски гиперкубов Q_n. В частности, доказано, что гиперкуб Q_n ($n \geq 3$) имеет интервальную тотальную раскраску тогда и только тогда, когда $n + 1 \leq t \leq \frac{(n+1)(n+2)}{2}$.