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On pre-Hamiltonian Cycles in Hamiltonian Digraphs

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Abstract

Let D be a strongly connected directed graph of order $n \geq 4$. In [14] (J. of Graph Theory, Vol.16, No. 5, 51-59, 1992) Y. Manoussakis proved the following theorem: Suppose that D satisfies the following condition for every triple x, y, z of vertices such that x and y are nonadjacent: If there is no arc from x to z, then $d(x)+d(y)+d^+(x)+d^-(z) \geq 3n-2$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2$. Then D is Hamiltonian. In this paper we show that: If D satisfies the condition of Manoussakis' theorem, then D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph with partite sets of cardinalities $n/2$ and $n/2$.

Keywords: Digraphs, Cycles, Hamiltonian cycles, Pre-Hamiltonian cycles, Longest non-Hamiltonian cycles.

1. Introduction

A directed graph (digraph) D is Hamiltonian if it contains a Hamiltonian cycle, i.e., a cycle of length n, and is pancyclic if it contains cycles of all lengths m, $3 \leq m \leq n$, where n is the number of vertices in D. We recall the following well-known degree conditions (Theorems 1.1-1.8) which guarantee that a digraph is Hamiltonian. In each of the conditions (Theorems 1.1-1.8) below D is a strongly connected digraph of order n :

Theorem 1.1: (Ghouila-Houri [12]). *If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.2: (Woodall [18]). *If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.3: (Meyniel [15]). *If $n \geq 2$ and $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

It is easy to see that Meyniel's theorem is a common generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.3, see [5].

C. Thomassen [17] (for $n = 2k+1$) and S. Darbinyan [7] (for $n = 2k$) proved the following:

Theorem 1.4: (C. Thomassen [17], S. Darbinyan [7]). *If D is a digraph of order $n \geq 5$ with minimum degree at least $n-1$ and with minimum semi-degree at least $n/2-1$, then D is Hamiltonian (unless some extremal cases which are characterized).*

For the next theorem we need the following:
Definition 1: ([14]). *Let k be an arbitrary nonnegative integer. A digraph D satisfies the condition A_k if and only if for every triple x, y, z of vertices such that x and y are nonadjacent: If there is no arc from x to z, then $d(x)+d(y)+d^+(x)+d^-(z) \geq 3n-2+k$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2+k$.*

Theorem 1.5: (Y. Manoussakis [14]). *If a digraph D of order $n \geq 4$ satisfies the condition A_0, then D is Hamiltonian.*

Each of these theorems imposes a degree condition on all pairs of nonadjacent vertices (or on all vertices). The following three theorems impose a degree condition only for some pairs of nonadjacent vertices.

Theorem 1.6: (Bang-Jensen, Gutin, H.Li [2]). *Suppose that $min\{d(x), d(y)\} \geq n-1$ and $d(x)+d(y) \geq 2n-1$ for any pair of nonadjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.*

Theorem 1.7: (Bang-Jensen, Gutin, H.Li [2]).*Suppose that $min\{d^+(x)+d^-(y), d^-(x)+d^+(y)\} \geq n$ for any pair of nonadjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Theorem 1.8: (Bang-Jensen, Guo, Yeo [3]). *Suppose that $d(x)+d(y) \geq 2n-1$ and $min\{d^+(x)+d^-(y), d^-(x)+d^+(y)\} \geq n-1$ for any pair of nonadjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Note that Theorem 1.8 generalizes Theorem 1.7.

In [11, 16, 6, 8] it was shown that if a digraph D satisfies the condition of one of Theorems 1.1, 1.2, 1.3 and 1.4, respectively, then D also is pancyclic (unless some extremal cases which are characterized). It is natural to set the following problem:

Characterize those digraphs which satisfy the conditions of Theorem 1.6 (1.7, 1.8) but are not pancyclic.

In many papers (in the mentioned papers as well), the existence of a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) is essential to the show that a given digraph (graph) is pancyclic or not. This indicates that the existence of a pre-Hamiltonian cycle in a digraph (graph) in a sense makes the pancyclic problem significantly easier. For the digraphs which satisfy the conditions of Theorem 1.6 or 1.7 or 1.8 in [9] and [10] the following results are proved:
(i) if the minimum semi-degree of a digraph D at least two and D satisfies the conditions of Theorem 1.6 or a digraph D is not a directed cycle and satisfies the conditions of Theorem 1.7, then either D contains a pre-Hamiltonian cycle (i.e., a cycle of length $n-1$) or n is even and D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$ or to the complete bipartite digraph $K^*_{n/2,n/2}$ minus one arc
(ii) if a digraph D is not a directed cycle and satisfies the conditions of Theorem 1.8, then

D contains a pre-Hamiltonian cycle or a cycle of length $n-2$.

In [14] the following conjecture was proposed:

Conjecture 1.9: *Any strongly connected digraph satisfying the condition A_3 is pancyclic.*

In this paper using some claims of the proof of Theorem 1.5 (see [14]) we prove the following theorem:

Theorem 1.10: *Any strongly connected digraph D on $n \geq 4$ vertices satisfying the condition A_0 contains a pre-Hamiltonian cycle or n is even and D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$.*

The following examples show the sharpness of the bound $3n-2$ in the theorem. The digraph consisting of the disjoint union of two complete digraphs with one common vertex or the digraph obtained from a complete bipartite digraph after deleting one arc show that the bound $3n-2$ in the above theorem is best possible.

2. Terminology and Notations

We shall assume that the reader is familiar with the standard terminology on the directed graphs (digraph) and refer the reader to [1] for terminology not discussed here. In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The order of D is the number of its vertices. Often we will write D instead of $A(D)$ and $V(D)$. The arc of a digraph D directed from x to y is denoted by xy. For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A,B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \rightarrow B$. The out-neighborhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighborhood of x. Similarly, if $A \subseteq V(D)$, then $N^+(x,A) = \{y \in A/xy \in A(D)\}$ and $N^-(x,A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x,A) = |N^+(x,A)|$ and $d^-(x,A) = |N^-(x,A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x,A) = d^+(x,A) + d^-(x,A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $\langle A \rangle$. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. For a cycle $C_k := x_1x_2 \cdots x_kx_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod k$. A cycle that contains all the vertices of D (respectively, all the vertices of D except one) is a Hamiltonian cycle (respectively, is a pre-Hamiltonian cycle). The concept of the pre-Hamiltonian cycle was given in [13]. If P is a path containing a subpath from x to y we let $P[x,y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x,y]$ denotes the subpath of C from x to y. A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. For an undirected graph G, we denote by G^* the symmetric digraph obtained from G by replacing every edge xy with the pair xy, yx of arcs. $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinalities p and q. Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b. Let C be a non-Hamiltonian cycle in digraph D. An (x, y)-path P is a C-bypass if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$.

3. Preliminaries

The following well-known simple Lemmas 3.1-3.4 are the basis of our results and other theorems on directed cycles and paths in digraphs. They will be used extensively in the proofs of our results.

Lemma 3.1: [11]. *Let D be a digraph of order $n \geq 3$ containing a cycle C_m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, C_m) \geq m + 1$, then D contains a cycle C_k for all $k \in [2, m+1]$.*

The following lemma is a slight modification of the lemma by Bondy and Tomassen [5].

Lemma 3.2: *Let D be a digraph of order $n \geq 3$ containing a path $P := x_1 x_2 \dots x_m$, $m \in [2, n-1]$ and let x be a vertex not contained in this path. If one of the following conditions holds:*

(i) $d(x, P) \geq m + 2$;

(ii) $d(x, P) \geq m + 1$ and $xx_1 \notin D$ or $x_m x \notin D$;

(iii) $d(x, P) \geq m$, $xx_1 \notin D$ and $x_m x \notin D$, then there is an $i \in [1, m-1]$ such that $x_i x, xx_{i+1} \in D$, i.e., D contains a path $x_1 x_2 \dots x_i x x_{i+1} \dots x_m$ of length m (we say that x can be inserted into P or the path $x_1 x_2 \dots x_i x x_{i+1} \dots x_m$ is an extended path from P with x).

If in Lemmas 3.1 and 3.2 instead of the vertex x consider a path Q, then we get the following Lemmas 3.3 and 3.4, respectively.

Lemma 3.3: *Let $C_k := x_1 x_2 \dots x_k x_1$, $k \geq 2$, be a non-Hamiltonian cycle in a digraph D. Moreover, assume that there exists a path $Q := y_1 y_2 \dots y_r$, $r \geq 1$, in $D - C_k$. If $d^-(y_1, C_k) + d^+(y_r, C_k) \geq k + 1$, then for all $m \in [r+1, k+r]$ the digraph D contains a cycle C_m of length m with vertex set $V(C_m) \subseteq V(C_k) \cup V(Q)$.*

Lemma 3.4: *Let $P := x_1 x_2 \dots x_k$, $k \geq 2$, be a non-Hamiltonian path in a digraph D. Moreover, assume that there exists a path $Q := y_1 y_2 \dots y_r$, $r \geq 1$, in $D - P$. If $d^-(y_1, P) + d^+(y_r, P) \geq k + d^-(y_1, \{x_k\}) + d^+(y_r, \{x_1\})$, then D contains a path from x_1 to x_k with vertex set $V(P) \cup V(Q)$.*

For the proof of our result we also need the following:

Lemma 3.5: ([14]). *Let D be a digraph on $n \geq 3$ vertices satisfying the condition A_0. Assume that there are two distinct pairs of nonadjacent vertices x, y and x, z in D. Then either $d(x) + d(y) \geq 2n - 1$ or $d(x) + d(z) \geq 2n - 1$.*

4. The Proof of Theorem 1.10

In the proof of Theorem 1.10 we often will use the following definition:

Definition 2: *Let $P_0 := x_1x_2 \dots x_m$, $m \geq 2$, be an arbitrary (x_1, x_m)-path in a digraph D and let $y_1, y_2, \dots y_k \in V(D) - V(P_0)$. For $i \in [1, k]$ we denote by P_i an (x_1, x_m)-path in D with vertex set $V(P_{i-1}) \cup \{y_j\}$ (if it exists) such that P_i is an extended path obtained from P_{i-1} with some vertex y_j, where $y_j \notin V(P_{i-1})$. If $e+1$ is the maximum possible number of these paths $P_0, P_1, \dots, P_e$, $e \in [0, k]$, then we say that P_e is an extended path obtained from P_0 with vertices $y_1, y_2, \dots, y_k$ as much as possible. Notice that P_i ($i \in [0, e]$) is an (x_1, x_m)-path of length $m + i - 1$.*

Proof of Theorem 1.10: Let $C := x_1x_2 \dots x_kx_1$ be a longest non-Hamiltonian cycle in D of length k, and let C be chosen so that $\langle V(D) - V(C)\rangle$ has the minimum number of connected components. Suppose that $k \leq n-2$ and $n \geq 5$ (the case $n = 4$ is trivial). It is easy to show that $k \geq 3$. We will prove that D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$. Put $R := V(D) - V(C)$. Let $R_1, R_2, \dots, R_q$ be the connected components of $\langle R\rangle$ (i.e., if $q \geq 2$, then for any pair i, j, $i \neq j$, there is no arc between R_i and R_j). In [14] it was proved that for any R_i, $i \in [1, q]$, the subdigraph $\langle V(C) \cup V(R_i)\rangle$ contains a C-bypass. (The existence of a C-bypass also follows from Bypass Lemma (see [4]), since $\langle V(C) \cup V(R_i)\rangle$ is strong and the condition A_0 implies that the underlying graph of the subdigraph $\langle V(C) \cup V(R_i)\rangle$ is 2-connected). Let $P := x_my_1y_2 \dots y_{t_i}x_{m+\lambda_i}$ be a C-bypass in $\langle V(C) \cup V(R_i)\rangle$ ($i \in [1, q]$ is arbitrary) and λ_i is considered to be minimum in the sense that there is no C-bypass $x_au_1u_2 \dots u_{l_i}x_{a+r_i}$ in $\langle V(C) \cup V(R_i)\rangle$ such that $r_i < \lambda_i$ and $\{x_a, x_{a+r_i}\}$ is a subset of $\{x_m, x_{m+1}, \dots, x_{m+\lambda_i}\}$.

We will distinguish two cases, according as there is a λ_i, $i \in [1, q]$, such that $\lambda_i = 1$ or not.

Assume first that $\lambda_i \geq 2$ for all $i \in [1, q]$. For this case one can show that (the proofs are the same as the proofs of Case 1, Lemma 2.3 and Claim 1 in [14]) if $\lambda_i \geq 2$, then $t_i = |R_i| = 1$, in $\langle V(C)\rangle$ there is an $(x_{m+\lambda_i}, x_m)$-path (say, P') of length $k-2$ with vertex set $V(P') = V(C) - \{z_i\}$, where $z_i \in \{x_{m+1}, x_{m+2}, \dots, x_{m+\lambda_i-1}\}$ and $d(y_1) + d(z_i) \leq 2n-2$ (note that y_1 and z_i are nonadjacent). From $|R| \geq 2$ and $|R_i| = 1$ (for all i) it follows that $q \geq 2$. If $u \in R_2$, then $d(u) = d(u, C) \leq k$ (by Lemma 3.1) and $d(z_1, R) = 0$ (by minimality of q), in particular, the vertices z_1 and u are nonadjacent. Therefore, $d(z_1) = d(z_1, C) \leq k$ and $d(z_1) + d(u) \leq 2n-2$. This in connection with $d(y_1) + d(z_1) \leq 2n-2$ contradicts Lemma 3.5.

Assume second that $\lambda_i = 1$ for all $i \in [1, q]$. It is clear that $q = 1$. Put $t := t_1$ and $\lambda := \lambda_1 = 1$.

Observe that if $v_1v_2 \dots v_j$ (maybe, $j = 1$) is a path in $\langle R\rangle$ and $x_iv_1 \in D$, then $v_jx_{i+j} \notin D$ since C is the longest non-Hamiltonian cycle in D and $k \leq n-2$. We shall use this often, without mentioning this explicitly.

The following claim follows immediately from $\lambda = 1$ and the maximality of C.

Claim 1: *$R = \{y_1, y_2, \dots, y_t\}$ (i.e., $t = n-k \geq 2$), $y_1y_2 \dots y_t$ is a Hamiltonian path in $\langle R\rangle$ and if $1 \leq i < j-1 \leq t-1$, then $y_iy_j \notin D$.*

From Claim 1 it follows that

$$d^+(y_1, R) = d^-(y_t, R) = 1 \quad \text{and if} \quad i \in [1, t-1], \text{ then } d^+(y_i, R) \leq i; \tag{1}$$

$$d(y_1,R),\ d(y_t,R)\leq n-k \quad \text{and if} \quad i\in[2,t-1], \quad \text{then} \quad d(y_i,R)\leq n-k+1. \qquad (2)$$

Claim 2: *(i). If $x_iy_1\in D$, then $d^-(x_{i+1},\{y_1,y_2,\ldots,y_{t-1}\})=0$;*
(ii). If $y_tx_{i+1}\in D$, then $d^+(x_i,\{y_2,y_3,\ldots,y_t\})=d^+(x_{i-1},\{y_1,y_2,\ldots,y_{t-1}\})=0$;
(iii). $d(y_1,C)\leq k$, $d(y_t,C)\leq k$ and $d(y_j,C)\leq k-1$ for all $j\in[2,t-1]$ (by Lemma 3.2(iii) and Claim 2(ii) since $\lambda=1$). □

Claim 3: *Assume that $\langle R\rangle$ is strong. If $d^+(x_i,R)\geq 1$, $d^-(x_j,R)\geq 1$ and $|C[x_i,x_j]|\geq 3$ for some two distinct vertices x_i,x_j ($i,j\in[1,k]$), then the following holds:*
(i) $d^-(x_{j-1},R)\neq 0$ or $A(R,C[x_{i+1},x_{j-2}])\neq\emptyset$;
(ii) $d^+(x_{i+1},R)\neq 0$) or $A(R,C[x_{i+2},x_{j-1}])\neq\emptyset$.
(Here if $|C[x_i,x_j]|=3$, then $C[x_{i+1},x_{j-2}]=\emptyset$ and $C[x_{i+2},x_{j-1}]=\emptyset$).

Proof of Claim 3: Suppose that Claim 3(i) is false. Without loss of generality, assume that $x_ky_f, y_gx_l\in D$ ($l\in[2,k-1]$)

$$d^-(x_{l-1},R)=0 \quad \text{and} \quad A(R,C[x_1,x_{l-2}])=\emptyset. \qquad (3)$$

The subdigraph $\langle R\rangle$ contains a (y_f,y_g)-path (say $P(y_f,y_g)$) since R is strong. We extend the path $P_0:=C[x_l,x_k]$ with the vertices $x_1,x_2,\ldots,x_{l-1}$ as much as possible. Then some vertices $z_1,z_2,\ldots,x_d\in\{x_1,x_2,\ldots,x_{l-1}\}$, $d\in[1,l-1]$, are not on the extended path P_e (for otherwise, it is not difficult to see that by Definition 2 there is an (x_l,x_k)-path P_i, $i\in[0,e]$, which together with the path $P(y_f,y_g)$ and the arcs x_ky_f, y_gx_l forms a non-Hamiltonian cycle longer than C). Therefore, by Lemma 3.2(i), for all $s\in[1,d]$ the following holds

$$d(z_s,C)\leq k+d-1. \qquad (4)$$

From (3) it follows that $y_1x_{l-1}\notin D$ and $y_tx_{l-1}\notin D$. Hence, by Lemma 3.2(ii), we have

$$d(y_1,C)\leq k-l+2 \quad \text{and} \quad d(y_t,C)\leq k-l+2$$

since neither y_1 nor y_t cannot be inserted into $C[x_{l-1},x_k]$. This together with (2) implies that

$$d(y_1)\leq n-l+2 \quad \text{and} \quad d(y_t)\leq n-l+2. \qquad (5)$$

If there exists a z_s such that $d(z_s,R)=0$, then by $d\leq l-1$, (4) and (5) we obtain that

$$d(z_s)+d(y_1)\leq 2n-2 \quad \text{and} \quad d(z_s)+d(y_t)\leq 2n-2,$$

which contradicts Lemma 3.5 since z_s,y_1 and z_s,y_t are two distinct pairs of nonadjacent vertices. Assume, therefore, that there is no z_s such that $d(z_s,R)=0$. Then from (3) it follows that $d=1$, $z_1=x_{l-1}$ and $d^+(x_{l-1},R)\geq 1$. Therefore, D contains an (x_l,x_k)-path, say Q, with vertex set $V(C)-\{x_{l-1}\}$. Since $\langle R\rangle$ is strong, it follows that in $\langle R\rangle$ there is a (y_f,y_g)-path, say T. This path T together with the path Q and the arcs x_ky_f, y_gx_l forms a cycle C' which does not contain x_{l-1}. From the maximality of C it follows that $|T|=1$ (i.e., $y_f=y_g$) and

$$d^+(x_k,R-\{y_f\})=d^-(x_l,R-\{y_f\})=0. \qquad (6)$$

So, the cycle C' has the length k and $V(C')=V(C)\cup\{y_f\}-\{x_{l-1}\}$. It is not difficult to see that the vertices x_{l-1}, y_f are nonadjacent (for otherwise $x_{l-1}y_f\in D$ and $x_{l-1}y_fx_l\ldots x_kx_1\ldots x_{l-1}$ is a cycle of length $k+1$, a contradiction). From this and

$d^-(x_{l-1}, R) = 0$ (by (3)) we have $d(x_{l-1}, R) \le n-k-1$. This together with $d = 1$ and (4) implies that $d(x_{l-1}) \le n-1$.

Assume first that $y_f \ne y_1$.

Let $x_{l-1}y_1 \in D$. Then $y_f = y_t$ (by Claim 2(i)) and for the triple of vertices y_t, x_{l-1}, y_1 condition A_0 holds, since $y_1x_{l-1} \notin D$ and y_t, x_{l-1} are nonadjacent. Since $y_tx_l \in D$, from (3) and Claim 2(ii) it follows that $d(x_{l-1}, R-\{y_1\}) = 0$, i.e., $d(x_{l-1}, R) = 1$. This together with (4) and $d = 1$ gives $d(x_{l-1}) \le k+1$. Since D contains no cycle of length $k+1$, it follows that for the arc $x_{l-1}y_1$ and the cycle C', by Lemma 3.3, the following holds $d^-(x_{l-1}, C') + d^+(y_1, C') \le k$. This together with $d^+(y_1, R) = 1$ and $d^-(x_{l-1}, R) = 0$ implies that $d^-(x_{l-1}) + d^+(y_1) \le n-2$ (here we consider the cases $k = n-2$ and $k \le n-3$ separately). Therefore, using condition A_0, (5), $d(x_{l-1}) \le n-1$ and $l \ge 2$ we obtain

$$3n-2 \le d(y_t) + d(x_{l-1}) + d^-(x_{l-1}) + d^+(y_1) \le 3n-3,$$

a contradiction.

Let now $x_{l-1}y_1 \notin D$. Then by (3) the vertices x_{l-1}, y_1 are nonadjacent. From this $t \ge 3$ since y_f, x_{l-1} are nonadjacent and $d^+(x_{l-1}, R) \ge 1$. Thus, we have $x_ky_1 \notin D$, $y_1x_l \notin D$ (by (6)) and $d(y_1, C[x_1, x_{l-1}]) = 0$. Therefore, since y_1 cannot be inserted into $C[x_l, x_k]$, using Lemma 3.2(iii) and (2) we obtain $d(y_1) \le n-l$. Notice that (by (2) and (4))

$$d(x_{l-1}) = d(x_{l-1}, C) + d(x_{l-1}, R-\{y_1, y_f\}) \le k + d(x_{l-1}, R-\{y_1, y_f\}) \le n-2,$$

and (by Lemma 3.2(i) and $d(y_f, C[x_1, x_{l-1}]) = 0$),

$$d(y_f) = d(y_f, C) + d(y_f, R) \le k-l+2 + d(y_f, R).$$

From the last three inequalities we obtain that

$$d(y_1) + d(x_{l-1}) \le 2n-l-2,$$

and

$$d(y_f) + d(x_{l-1}) \le 2k-l+2 + d(x_{l-1}, R-\{y_1, y_f\}) + d(y_f, R).$$

Notice that

$$d(x_{l-1}, R-\{y_1, y_f\}) + d(y_f, R) \le n-k-2+n-k = 2n-2k-2$$

since if $x_{l-1}y_j \in D$, then $y_jy_f \notin D$, where $y_j \ne y_1, y_f$. The last two inequalities give $d(y_f) + d(x_{l-1}) \le 2n-l \le 2n-2$. This together with $d(y_1) + d(x_{l-1}) \le 2n-l-2$ contradicts Lemma 3.5 since x_{l-1}, y_1 and x_{l-1}, y_f are two distinct pairs of nonadjacent vertices.

Assume next that $y_f = y_1$. If x_{l-1}, y_t are nonadjacent, then $d(x_{l-1}, \{y_1, y_t\}) = 0$ and $d(x_{l-1}, R) \le n-k-2$. Hence, by (4) and $d = 1$ we have $d(x_{l-1}) \le n-2$. This together with (5) implies that

$$d(y_1) + d(x_{l-1}) \le 2n-2 \quad \text{and} \quad d(y_t) + d(x_{l-1}) \le 2n-2,$$

which contradicts Lemma 3.5, since y_1, x_{l-1} and y_t, x_{l-1} are two distinct pairs of nonadjacent vertices. So, we can assume that $x_{l-1}y_t \in D$. Since C' is a longest non-Hamiltonian cycle, $d^-(x_{l-1}, R) = 0$, (3) and $d^+(y_t, R-\{y_1\}) \le n-k-2$, from Lemma 3.3 it follows that

$d^-(x_{l-1}) + d^+(y_t) \leq n-2$. Now using (5), $d(x_{l-1}) \leq n-1$ and the condition A_0, for the triple of the vertices x_{l-1}, y_1, y_t we obtain

$$3n-2 \leq d(y_1) + d(x_{l-1}) + d^+(y_t) + d^-(x_{l-1}) \leq 3n-l-1 \leq 3n-3,$$

which is a contradiction. Claim 3 is proved.

In particular, from Claim 3 immediately follows the following

Claim 4: *Assume that $\langle R\rangle$ is strong and $d^+(x_i, R) \geq 1$, $d^-(x_j, R) \geq 1$ for some two distinct vertices x_i and x_j. Then the following holds:*

(i) if $|C[x_i, x_j]| \geq 3$, then $A(R, C[x_{i+1}, x_{j-1}]) \neq \emptyset$;

(ii) if $|C[x_i, x_j]| = 3$, then $d^+(x_{i+1}, R) \geq 1$ and $d^-(x_{j-1}, R) \geq 1$.

Now we divide the proof of the theorem into two parts: $k \leq n-3$ and $k = n-2$.

Part 1. $k \leq n-3$, i.e., $t \geq 3$.

For this part first we will prove the following Claims 5-10 below.

Claim 5: *Let $t \geq 3$ and $y_t y_1 \in D$. Then the following holds*

(i) if $x_i y_1 D$, then $d^-(x_{i+2}, R) = 0$; (ii) if $y_t x_i \in D$, then $d^+(x_{i-2}, R) = 0$, where $i \in [1, k]$.

Proof of Claim 5: (i). Suppose, on the contrary, that for some $i \in [1, k]$ $x_i y_1 \in D$ and $d^-(x_{i+2}, R) \neq 0$. Without loss of generality, we assume that $x_i = x_1$ and $d^-(x_3, R) \neq 0$. Then $d^-(x_3, R - \{y_1\}) = 0$ and $y_1 x_3 \in D$. It is easy to see that y_1, x_2 are nonadjacent and

$$d^-(x_2, \{y_1, y_2, \ldots, y_{t-1}\}) = d^+(x_2, \{y_1, y_3, y_4, \ldots, y_t\}) = 0, \quad \text{i.e.,} \quad d(x_2, R) \leq 2. \qquad (7)$$

Since neither y_1 nor x_2 can be inserted into $C[x_3, x_1]$, using (2), (7) and Lemma 3.2, we obtain that

$$d(y_1) = d(y_1, C) + d(y_1, R) \leq k + n - k = n \quad \text{and} \quad d(x_2) = d(x_2, C) + d(x_2, R) \leq k+2.$$

On the other hand, by Lemma 3.3 and (1) we have that $d^-(y_t) + d^+(y_1) \leq k+2$ since the arc $y_t y_1$ cannot be inserted into C. Therefore, by condition A_0, the following holds

$$3n-2 \leq d(y_1) + d(x_2) + d^-(y_t) + d^+(y_1) \leq n + 2k + 4,$$

since y_1, x_2 are nonadjacent and $y_1 y_t \notin D$. From this and $k \leq n-3$ it follows that $k = n-3$, $x_2 y_2, y_2 y_1 \in D$ and hence, the cycle $x_2 y_2 y_1 x_3 x_4 \ldots x_k x_1 x_2$ has length $k+2$. This contradicts the supposition that C is a maximal non-Hamiltonian cycle.

To show that (ii) is true, it is sufficient to apply the same arguments to the converse digraph of D. Claim 5 is proved.

Claim 6: *If $t \geq 3$ and the vertices y_1, y_t are nonadjacent, then $t = 3$ and $y_3 y_2$, $y_2 y_1 \in D$.*

Proof of Claim 6: Without loss of generality, we can assume that $x_1 y_1$, $y_t x_2 \in D$ (since $\lambda = 1$).

Assume first that $t \geq 4$ and $y_t y_i \in D$ for some $i \in [2, t-2]$. Since the arc $y_t y_i$ cannot be inserted into C, using Lemma 3.3, we obtain

$$d^-(y_t, C) + d^+(y_i, C) \leq k. \qquad (8)$$

From Claim 1 and the condition that y_1, y_t are nonadjacent it follows that

$$d(y_1, R) \leq n-k-1 \quad \text{and} \quad d(y_t, R) \leq n-k-1.$$

This together with Claim 2(iii) implies that $d(y_1)$ and $d(y_t) \leq n-1$. Since y_1, y_t are nonadjacent and $y_i y_t \notin D$, using (1), (8) and applying the condition A_0 to the triple of the vertices y_1, y_t, y_i, we obtain

$$3n-2 \leq d(y_1) + d(y_t) + d^-(y_t, C) + d^+(y_i, C) + d^-(y_t, R) + d^+(y_i, R) \leq 3n-3,$$

which is a contradiction.

Assume second that $t \geq 4$ and $y_t y_i \notin D$ for all $i \in [2, t-2]$. We also can assume that $y_i y_1 \notin D$ for all $i \in [3, t-1]$. Therefore, $d(y_1, R) \leq 2$ and $d(y_t, R) \leq 2$. This together with Claim 2(iii) implies that $d(y_1) \leq k+2$, $d(y_t) \leq k+2$ and hence

$$d(y_1) + d(y_t) \leq 2k+4. \tag{9}$$

From $t \geq 4$ and the above assumptions it follows that y_1, y_t and y_1, y_{t-1} are two distinct pairs of nonadjacent vertices. From (9) and $k \leq n-4$ it follows that $d(y_1)+d(y_t) \leq 2n-4$. On the other hand, since $d(y_1) \leq k+2$, $d(y_{t-1}, C) \leq k-1$ (by Claim 2(iii)) and $d(y_{t-1}, R) \leq n-k$ (by Claim 1), we have

$$d(y_1) + d(y_{t-1}) \leq 2n-3.$$

This together with $d(y_1)+d(y_t) \leq 2n-4$ contradicts Lemma 3.5. We, thus, proved that the case $t \geq 4$ is impossible.

Assume finally that $t = 3$. Now we will show that $y_3 y_2 \in D$. Assume that this is not the case, i.e., $y_3 y_2 \notin D$. Then we can apply the condition A_0 to the triple of the vertices y_1, y_3, y_2, since the vertices y_1, y_3 are nonadjacent and $y_3 y_2 \notin D$. Notice that the arc $y_2 y_3$ cannot be inserted into C and hence $d^-(y_2, C) + d^+(y_3, C) \leq k$ (by Lemma 3.3). Therefore, by A_0 and Claim 2(iii), we obtain

$$3n-2 \leq d(y_1) + d(y_3) + d^-(y_2) + d^+(y_3) \leq 3k+4 \leq 3n-5,$$

which is a contradiction. Therefore $y_3 y_2 \in D$.

Similarly we obtain a contradiction if we assume that $y_2 y_1 \notin D$. Therefore, $y_2 y_1 \in D$. Claim 6 is proved.

Claim 7: *If $t \geq 3$, then $y_t y_1 \in D$.*

Proof of Claim 7: Suppose, on the contrary, that $t \geq 3$ and $y_t y_1 \notin D$, i.e., y_1, y_t are nonadjacent. Then by Claim 6, $t = 3$ and $y_3 y_2, y_2 y_1 \in D$. Without loss of generality, assume that $x_1 y_1$ and $y_3 x_2 \in D$ (since $\lambda = 1$). Notice that $d(y_1), d(y_3) \leq n-1$ (by Lemma 3.1) and hence, $d(y_1) + d(y_3) \leq 2n-2$. We will distinguish two cases, according as there is an arc from R to $\{x_3, x_4, \dots, x_k\}$ or not.

Case 7.1. $A(R \to \{x_3, x_4, \dots, x_k\}) \neq \emptyset$.

Then there exists a vertex x_l with $l \in [3, k]$ such that $d^-(x_l, R) \geq 1$ and for $l \geq 4$, $A(R \to \{x_3, x_4, \dots, x_{l-1}\}) = \emptyset$.

If $l = 3$, then from $d^-(x_3, \{y_2, y_3\}) = 0$ it follows that $y_1 x_3 \in D$. From this it is easy to see that $d(x_2, \{y_1, y_2\}) = 0$. Since neither y_1 nor y_3 and x_2 can be inserted into $C[x_3, x_1]$ using Lemma 3.2 we obtain that $d(y_1)$, $d(y_3)$ and $d(x_2) \leq n-1$. Hence, $d(y_1)+d(y_3) \leq 2n-2$ and $d(y_1)+d(x_2) \leq 2n-2$, which contradicts Lemma 3.5 since y_1, y_3 and y_1, x_2 are two distinct pairs of nonadjacent vertices.

Assume, therefore, that $l \geq 4$. If $d^+(x_{l-1}, R) = 0$, then $d(x_{l-1}, R) = 0$ by minimality of l. Therefore, Claim 4 implies that there is no $x_i \in C[x_2, x_{l-2}]$ such that $d^+(x_i, R) \geq 1$. Therefore, by the minimality of l we have

$$A(R, C[x_3, x_{l-1}]) = \emptyset \quad \text{and} \quad d^+(x_2, R) = 0,$$

which contradicts Claim 3(ii) since $x_1y_1 \in D$ and $d^-(x_l, R) \geq 1$. Assume, therefore, that $d^+(x_{l-1}, R) \geq 1$. Without loss of generality, we may assume that $y_gx_l \in D$ and $x_{l-1}y_f \in D$. It is easy to see that $y_f \neq y_g$, $y_f, y_g \in \{y_1, y_3\}$ (i.e., $x_{l-1}y_2 \notin D$ and $y_2x_l \notin D$) and the vertices x_{l-1}, x_g are nonadjacent.

Assume first that $l = 4$. If $y_g = y_3$ (i.e., $y_3x_4 \in D$), then $x_1y_1y_2y_3x_4 \ldots x_{n-3}x_1$ is a cycle of length $n-1$, a contradiction. Assume, therefore, that $y_g = y_1$ and $y_f = y_3$, i.e., y_1x_4 and $x_3y_3 \in D$. Then the vertices x_2, y_2 are clearly nonadjacent and $x_2y_3 \notin D$. Since $y_1x_4 \in D$ and $d^-(x_3, R) = 0$, Claim 4(ii) implies that $x_2y_1 \notin D$. Therefore, $d(x_2, \{y_1, y_2\}) = 0$. Notice that x_2 cannot be inserted into the path $C[x_4, x_1]$ (for otherwise in D there is a cycle of length $n-3$ which does not contain the vertices y_2, y_3, x_3 but this contradicts Claim 6 since y_2, x_3 are nonadjacent and $y_3x_3 \notin D$). Now by Lemma 3.2 and the above observation we obtain that

$$d(x_2) = d(x_2, C[x_4, x_1]) + d(x_2, R) + d(x_2, \{x_3\}) \leq n - 1.$$

Therefore, $d(y_1) + d(x_2) \leq 2n - 2$, which together with $d(y_1) + d(y_3) \leq 2n - 2$ contradicts Lemma 3.5, since y_1, x_2 and y_1, y_3 are two distinct pairs of nonadjacent vertices.

Assume next that $l \geq 5$. From the minimality of l, $d^-(x_{l-1}, R) = 0$ and Claim 4(ii) it follows that $d(x_{l-2}, R) = 0$. Therefore, there is no $x_i \in C[x_2, x_{l-2}]$ such that $d^+(x_i, R) \geq 1$, in particular, $x_2y_3 \notin D$. Therefore

$$A(C[x_3, x_{l-2}], R) = \emptyset \quad \text{and} \quad d(x_2, R) = 1,$$

(only $y_3x_2 \in D$). Since $y_g \neq y_2$ and x_{l-1}, y_g are nonadjacent, we have $d(x_{l-1}, R) = 1$ (only $x_{l-1}y_{4-g} \in D$). By the above observation we have

$$d(y_1, C[x_2, x_{l-2}]) = d(y_3, C[x_3, x_{l-2}]) = 0. \tag{10}$$

Since y_1 cannot be inserted into C, $x_2y_3 \notin D$ and $d^-(x_{l-1}, R) = 0$, using (10) and Lemma 3.2 we obtain that $d(y_1, C) \leq k - l + 3$. This together with $d(y_1, R) = 2$ implies that $d(y_1) \leq k - l + 5$.

Now we extend the path $P_0 := C[x_l, x_1]$ with the vertices $x_2, x_3, \ldots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_d \in \{x_2, x_3, \ldots, x_{l-1}\}$, $d \in [1, l-2]$, are not on the extended path P_e. Therefore, $d(z_i, C) \leq k + d - 1$ and hence, $d(z_i) \leq k + d$ for all $i \in [1, d]$. Thus we have $d(y_1) + d(z_i) \leq 2n - 3$ and $d(y_3) + d(z_i) \leq 2n - 3$ since there is a vertex z_i which is not adjacent to y_1 or y_3. This together with $d(y_1) + d(y_3) \leq 2n - 2$ contradicts Lemma 3.5 since y_1, z_i (or y_3, z_i) and y_1, y_3 are two distinct pairs of nonadjacent vertices. In each case we have a contradiction. The discussion of Case 7.1 is completed.

Case 7.2. $A(R \to \{x_3, x_4, \ldots, x_k\}) = \emptyset$.

Without loss of generality, we may assume that $A(\{x_3, x_4, \ldots, x_k\} \to R) = \emptyset$ (for otherwise, we consider the converse digraph of D for which the considered Case 7.1 holds). Therefore $A(R, \{x_3, x_4, \ldots, x_k\}) = \emptyset$. In particular, x_k is not adjacent to the vertices y_1 and y_3. Notice that

$$d(y_1) = d(y_1, R) + d(y_1, C) \leq 2 + d(y_1, \{x_1, x_2\}) \leq 5,$$

$d(y_3) \leq 5$ and $d(x_k) = d(x_k, C) \leq 2n - 8$. Therefore $d(x_k) + d(y_1) \leq 2n - 3$ and $d(x_k) + d(y_3) \leq 2n - 3$, which contradicts Lemma 3.5. Claim 7 is proved. □

Claim 8: *If $t \geq 3$ and for some $i \in [1,k]$ x_iy_1, then $A(R \to C[x_{i+2}, x_{i-1}]) = \emptyset$.*

Proof of Claim 8: Suppose that the claim is not true. Without loss of generality, we may assume that $x_1y_1 \in D$ and $A(R \to \{x_3, x_4, \dots, x_k\}) \neq \emptyset$. Then there is a vertex x_l with $l \in [3,k]$ such that $d^-(x_l, R) \geq 1$ and if $l \geq 4$, then $A(R \to \{x_3, x_4, \dots, x_{l-1}\}) = \emptyset$. We have that $y_ty_1 \in D$ (by Claim 7). In particular, $y_ty_1 \in D$ implies that $\langle R \rangle$ is strong. On the other hand, by Claim 5(i), $d^-(x_3, R) = 0$ and hence, $l \geq 4$. From $x_1y_1 \in D$ it follows that there exists a vertex x_r with $r \in [1, l-1]$ such that $d^+(x_r, R) \geq 1$. Choose r with these properties as maximal as possible. Let x_ry_f and $y_gx_l \in D$. Notice that in $\langle R \rangle$ there is a (y_f, y_g)-path since $\langle R \rangle$ is strong. Using Claims 4(i) and 3(ii) we obtain that $r = l-1$. Then $y_f \neq y_g$ and in $\langle R \rangle$ any (y_f, y_g)-path is a Hamiltonian path. Since $\langle R \rangle$ is strong, from $d^-(x_{l-1}, R) = 0$, $d^-(x_l, R) \geq 1$ and from Claim 3(i) it follows that $A(\{x_2, x_3, \dots, x_{l-2}\} \to R) = \emptyset$, in particular, $d^+(x_2, R) = 0$. By the above observations we have

$$A(\{x_3, x_4, \dots, x_{l-2}\}, R) = \emptyset, \quad d(y_1, \{x_2, x_3, \dots, x_{l-2}\}) = d(x_2, \{y_1, y_2, \dots, y_{t-1}\}) = 0. \quad (11)$$

Note that x_2, y_1 and x_2, y_2 are two distinct pairs of nonadjacent vertices. We extend the path $P_0 := C[x_l, x_1]$ with the vertices $x_2, x_3, \dots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_2, x_3, \dots, x_{l-1}\}$, where $d \in [1, l-2]$, are not on the extended path P_e (for otherwise, since in $\langle R \rangle$ there is a (y_f, y_g)-path, using the path P_{e-1} or P_e we obtain a non-Hamiltonian cycle longer than C). By Lemma 3.2, for all $i \in [1,d]$ we have

$$d(z_i, C) \leq k + d - 1 \quad \text{and} \quad d(z_i) = d(z_i, C) + d(z_i, R) \leq k + d - 1 + d(z_i, R). \quad (12)$$

Assume that there is a vertex $z_i \neq x_{l-1}$. Then, by (11), $d(z_i, R) \leq 1$ (since $d(x_2, R) \leq 1$). Notice that y_1, z_i and y_2, z_i are two distinct pairs of nonadjacent vertices (by (11)). Since neither y_1 nor y_2 can be inserted into $C[x_{l-1}, x_1]$ and $y_1x_{l-1} \notin D$, $y_2x_{l-1} \notin D$, by Lemma 3.2(ii) and (11) for $j = 1$ and 2 we obtain

$$d(y_j, C) = d(y_j, C[x_{l-1}, x_1]) \leq k - l + 3. \quad (13)$$

In particular, by (2),

$$d(y_1) = d(y_1, C) + d(y_1, R) \leq k - l + 3 + n - k = n - l + 3.$$

This together with (12) and $d(z_i, R) \leq 1$ implies that

$$d(y_1) + d(z_i) \leq 2n - 2,$$

since $k \leq n-3$ and $d \leq l-2$. Therefore, by Lemma 3.5, $d(y_2) + d(z_i) \geq 2n-1$. Hence, by (2) and (12) we have

$$2n - 1 \leq d(y_2) + d(z_i) \leq n + d + d(z_i, R) + d(y_2, C).$$

From this, $d \leq l-2$ and (13) it follows that $d(y_2, C) = k - l + 3$, $d(z_i, R) = 1$ and $k = n-3$. Then $z_i = x_2$ and $y_tx_2 \in D$ (by (11) and $d^+(x_2, R) = 0$). Therefore, $x_1y_2 \notin D$. From this, $y_2x_{l-1} \notin D$ and $d(y_2, C) = k - l + 3$, by Lemma 3.2(iii), we conclude that y_2 can be inserted into C, which is contrary to our supposition that C is a longest non-Hamiltonian cycle.

Now assume that there is no $z_i \neq x_{l-1}$. Then $d = 1$, $z_1 = x_{l-1}$ and there is an (x_l, x_1)-path with vertex set $V(C) - \{x_{l-1}\}$. Therefore, $d^-(x_l, \{y_2, y_3, \dots, y_t\}) = 0$ (since $x_1y_1 \in D$)

and $y_1x_l \in D$. From this we have, $d(x_{l-1}, R - \{y_2\}) = 0$ since $y_ty_1 \in D$ and l is minimal, in particular, the vertices y_t, x_{l-1} are nonadjacent. This together with (12) implies that $d(x_{l-1}) \leq k+1$ (only $x_{l-1}y_2 \in D$ is possible). Notice that neither y_t nor the arc y_ty_1 can be inserted into C, and therefore, by Lemmas 3.2, 3.3 and by (1), (2) we obtain that $d(y_t) \leq n$ and $d^-(y_t) + d^+(y_1) \leq k+2$. Since $y_1y_t \notin D$ and y_t, x_{l-1} are nonadjacent we have that the triple of the vertices y_t, x_{l-1}, y_1 satisfies condition A_0. Therefore

$$3n - 2 \leq d(x_{l-1}) + d(y_t) + d^-(y_t) + d^+(y_1) \leq 3n - 3$$

since $k \leq n-3$, which is a contradiction. Claim 8 is proved. □

Claim 9: *If $t \geq 3$, x_1y_1 and $y_tx_2 \in D$, then $d^-(x_1, R) = 0$.*

Proof of Claim 9: Assume that $d^-(x_1, R) \geq 1$. By Claim 7, $y_ty_1 \in D$. Now using Claims 5(ii) and 8, we obtain that $d^+(x_k, R) = 0$ and

$$A(R \to \{x_3, x_4, \ldots, x_k\}) = \emptyset. \tag{14}$$

In particular, $d(x_k, R) = 0$. This together with $d^-(x_1, R) \geq 1$, (14) and Claim 3 implies that $A(\{x_2, x_3, \ldots, x_{k-1}\} \to R) = \emptyset$. Now again using (14) we get that $A(\{x_3, x_4, \ldots, x_k\}, R) = \emptyset$. This together with $d^+(x_2, R) = d^-(x_2, \{y_1, y_2, \ldots, y_{t-1}\}) = 0$ implies that $d(x_2, R) = 1$, $d(y_2, C) \leq 1$ (only $y_2x_1 \in D$ is possible) and $d(x_3, R) = 0$. Therefore, by (2),

$$d(y_2) + d(x_3) = d(y_2, C) + d(y_2, R) + d(x_3, R) + d(x_3, C) \leq n + k \leq 2n - 3$$

and $d(y_2) + d(x_2) \leq 2n - 2$, which contradicts Lemma 3.5 since y_2, x_3 and y_2, x_2 are two distinct pairs of nonadjacent vertices. This completes the proof of Claim 9. □

Claim 10: *If $t \geq 3$, x_1y_1 and $y_tx_2 \in D$, then $A(\{x_3, x_4, \ldots, x_k\} \to R) = \emptyset$.*

Proof of Claim 10: By Claim 7, $y_ty_1 \in D$. Suppose that $A(\{x_3, x_4, \ldots, x_k\} \to R) \neq \emptyset$. Recall that Claim 5(ii) implies that $d^+(x_k, R) = 0$. Let x_r, $r \in [3, k-1]$, be chosen so that $x_ry_i \in D$ for some $i \in [1, t]$ and r is maximum possible. Then $A(\{x_{r+1}, x_{r+2}, \ldots, x_k\}, R) = \emptyset$ and $d^-(x_1, R) = 0$ by Claim 8 and Claim 9, respectively. This together with $y_tx_2 \in D$ contradicts Claim 3(i). Claim 10 is proved.

Now we are ready to complete the proof of Theorem 1.10 for Part 1 (when $k \leq n-3$, i.e., $t \geq 3$). By Claim 7, $y_ty_1 \in D$. Without loss of generality, we may assume that x_1y_1 and $y_tx_2 \in D$ since $\lambda = 1$. Then from Claims 8, 9 and 10 it follows that

$$A(R \to \{x_3, x_4, \ldots, x_k, x_1\}) = A(\{x_3, x_4, \ldots, x_k\} \to R) = \emptyset.$$

From this and

$$d^-(x_2, \{y_1, y_2, \ldots, y_{t-1}\}) = d^+(x_1, \{y_2, y_3, \ldots, y_t\}) = 0$$

we obtain that x_1, y_2 and x_1, y_t are two distinct pairs of nonadjacent vertices and $d(y_2, C) \leq 1$, $d(y_t, C) \leq 2$, $d(x_1, R) = 1$. Therefore, $d(y_2) \leq n - k + 2$, $d(y_t) \leq n - k + 2$ (by (2)) and $d(x_1) \leq 2k - 1$. The last three inequalities imply that $d(y_2) + d(x_1) \leq 2n - 2$ and $d(y_t) + d(x_1) \leq 2n - 2$, which contradicts Lemma 3.5 and completes the discussion of Part 1.

Part 2. $k = n - 2$, i.e., $t = 2$.

For this part first we will prove Claims 11-16 below.

Claim 11: *If $x_iy_f \in D$ and $y_2y_1 \notin D$, where $i \in [1, n-2]$ and $f \in [1, 2]$, then there is no $l \in [3, n-2]$ such that $y_fx_{i+l-1} \in D$ and $d(y_f, \{x_{i+1}, x_{i+2}, \dots, x_{i+l-2}\}) = 0$.*

Proof of Claim 11: The proof is by contradiction. Suppose that $x_iy_f, y_fx_{i+l-1} \in D$ and $d(y_f, \{x_{i+1}, x_{i+2}, \dots, x_{i+l-2}\}) = 0$ for some $l \in [3, n-2]$. Without loss of generality, we may assume that $x_i = x_1$. Then x_1y_f, $y_fx_l \in D$ and $d(y_f, \{x_2, x_3, \dots, x_{l-1}\}) = 0$. Since D contains no cycle of length $n-1$, using Lemmas 3.2 and 3.3, we obtain that

$$d^-(y_1) + d^+(y_2) \le n-2 \quad \text{and} \quad d(y_f) \le n-l+2. \tag{15}$$

We extend the path $P_0 := C[x_l, x_1]$ with the vertices $x_2, x_3, \dots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \dots, z_d \in \{x_2, x_3, \dots, x_{l-1}\}$, $d \in [1, l-2]$, are not on the extended path P_e. Therefore, by Lemma 3.2, $d(z_1) = d(z_1, C) + d(z_1, \{y_{3-f}\}) \le n+d-1$. Now, since the vertices y_f, z_1 are nonadjacent and $y_2y_1 \notin D$, by condition A_0 and (15) we have

$$3n-2 \le d(y_f) + d(z_1) + d^-(y_1) + d^+(y_2) \le 3n-3,$$

a contradiction. Claim 11 is proved. □

Claim 12: *$y_2y_1 \in D$ (i.e., if $k = n-2$, then $\langle V(D) - V(C)\rangle$ is strong).*

Proof of Claim 12: Suppose, on the contrary, that $y_2y_1 \notin D$. Without loss of generality, we may assume that $x_1y_1 \in D$ and the vertices y_1, x_2 are nonadjacent. Then $y_2x_3 \notin D$ and since D contains no cycle of length $n-1$, using Lemma 3.3 for the arc y_1y_2 we obtain that

$$d^-(y_1) + d^+(y_2) \le n-2. \tag{16}$$

Case 12.1. $d^+(y_1, C[x_3, x_{n-2}]) \ge 1$.

Let x_l, $l \in [3, n-2]$, be chosen so that $y_1x_l \in D$ and l is minimum, i.e., $d^+(y_1, C[x_2, x_{l-1}]) = 0$. It is easy to see that the vertices y_1 and x_{l-1} are nonadjacent. By Claim 11, we can assume that $l \ge 5$ (if $l \le 4$, then $d(y_1, C[x_2, x_{l-1}]) = 0$, a contradiction to Claim 11) and $d^-(y_1, C[x_3, x_{l-2}]) \ge 1$. It follows that there exists a vertex x_r with $r \in [3, l-2]$ such that $x_ry_1 \in D$ and $d(y_1, C[x_{r+1}, x_{l-1}]) = 0$. Consequently, for the vertices y_1, x_r and x_l Claim 11 is not true, a contradiction.

Case 12.2. $d^+(y_1, C[x_3, x_{n-2}]) = 0$.

Then $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and either $y_1x_1 \in D$ or $y_1x_1 \notin D$.

Subcase 12.2.1. $y_1x_1 \in D$.

Then $x_{n-2}y_1 \notin D$ and hence, the vertices y_1, x_{n-2} are nonadjacent. Therefore, the triple of the vertices y_1, x_{n-2}, y_2 satisfies the condition A_0. Claim 11 implies that $d^-(y_1, C[x_2, x_{n-2}]) = 0$. This together with $d^+(y_1, C[x_2, x_{n-2}]) = 0$ and $y_2y_1 \notin D$ gives $d(y_1) = 3$. Clearly, $d(x_2) \le 2n-4$ and hence, for the vertices y_1, y_2, x_2 by condition A_0 and (16) we have,

$$3n-2 \le d(y_1) + d(x_2) + d^-(y_1) + d^+(y_2) \le 3n-3,$$

which is a contradiction.

Subcase 12.2.2. $y_1x_1 \notin D$.

Then $d^+(y_1, C) = 0$, $d^+(y_1) = 1$ and $d^+(y_2, C) \ge 1$ since D is strong. Without loss of generality, we may assume that $d^-(y_2, C) = 0$ (for otherwise for the vertex y_2 in the converse digraph of D we would have the above considered Case 12.1 or Subcase 12.2.1). Since the triple of the vertices y_1, y_2, x_3 satisfies the condition A_0, $d(y_1) \leq n-2$, $d(x_2) \leq 2n-5$ and (16), it is not difficult to show that $n \geq 7$.

Suppose first that $y_2x_2 \in D$. Then $x_{n-2}y_1 \notin D$ and hence, the vertices x_{n-2}, y_1 are nonadjacent.

Let for some $l \in [3, n-3]$ $x_ly_1 \in D$ and $d^-(y_1, C[x_{l+1}, x_{n-2}]) = 0$. Then $d(y_1, C[x_{l+1}, x_{n-2}]) = 0$ and $d(y_1) \leq l$ since $d^+(y_1, C) = 0$ and x_2, y_1 are nonadjacent. Extend the path $P_0 := C[x_2, x_l]$ with the vertices $x_{l+1}, x_{l+2}, \ldots, x_{n-2}, x_1$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_d \in \{x_{l+1}, x_{l+2}, \ldots, x_{n-2}, x_1\}$, $d \in [2, n-l-1]$, are not on the extended path P_e. For a vertex $z_i \neq x_1$ by Lemma 3.2 we obtain that $d(z_i) = d(z_i, C) + d(z_i, \{y_2\}) \leq n+d-1$. Therefore, since $y_2y_1 \notin D$ and the vertices z_i, y_1 are nonadjacent, by condition A_0 and (16), we get that

$$3n-2 \leq d(y_1) + d(z_i) + d^-(y_1) + d^+(y_2) \leq 3n-4,$$

which is a contradiction.

Let now $x_ly_1 \notin D$ for all $l \in [3, n-2]$, i.e., $d^-(y_1, C[x_3, x_{n-2}]) = 0$. Then from $d^+(y_1, C[x_2, x_{n-2}]) = 0$, $y_1x_1 \notin D$ and $x_{n-2}y_2 \notin D$ (since $d^-(y_2, C) = 0$) it follows that $d(y_1) = 2$ and $d(x_{n-2}) \leq 2n-5$. From this, since the vertices y_1, x_{n-2} are nonadjacent and $y_2y_1 \notin D$, by condition A_0 and (16) we have that

$$3n-2 \leq d(y_1) + d(x_{n-2}) + d^-(y_1) + d^+(y_2) \leq 3n-5,$$

which is a contradiction.

Suppose next that $y_2x_2 \notin D$. Then $d(y_2, \{x_2, x_3\}) = 0$, since $d^-(y_2, C) = 0$.

Let for some $l \in [4, n-2]$ $y_2x_l \in D$ and $d^+(y_2, C[x_2, x_{l-1}]) = 0$. Then $d(y_2, C[x_2, x_{l-1}]) = 0$ and the vertices y_1, x_{l-2} are nonadjacent since $d^+(y_1, C[x_2, x_{n-2}]) = 0$. It is easy to see that there exists a vertex $x_r \in \{x_1, x_2, \ldots, x_{l-3}\}$ such that $x_ry_1 \in D$ and $d(y_1, C[x_{r+1}, x_{l-2}]) = 0$. Thus, we have that $A(R, C[x_{r+1}, x_{l-2}]) = \emptyset$. Notice that $d(y_2) \leq n-l+1$ since $d^-(y_2, C) = 0$ and $d(y_2, C[x_2, x_{l-1}]) = 0$. We extend the path $P_0 := C[x_l, x_r]$ with the vertices $x_{r+1}, x_{r+2}, \ldots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_d \in \{x_{r+1}, x_{r+2}, \ldots, x_{l-1}\}$, $d \in [2, l-r-1]$, are not on the extended path P_e. Therefore, by Lemma 3.2 for $z_i \neq x_{l-1}$ we have, $d(z_i) \leq n+d-3$. Now by condition A_0 and (16) we obtain

$$3n-2 \leq d(y_2) + d(z_i) + d^-(y_1) + d^+(y_2) < 3n-3,$$

a contradiction.

Let now $d^+(y_2, \{x_2, x_3, \ldots, x_{n-2}\}) = 0$. Then $d(y_2) = 2$, $d(x_2) \leq 2n-6$ and the vertices x_2, y_2 are nonadjacent. By condition A_0 we have

$$3n-2 \leq d(y_2) + d(x_2) + d^-(y_1) + d^+(y_2) < 3n-3,$$

a contradiction. Claim 12 is proved. □

Claim 13: *For any $i \in [1, n-2]$ and $f \in [1, 2]$ the following holds*
i) $d^-(y_f, \{x_{i-1}, x_i\}) \leq 1$ and ii) $d^+(y_f, \{x_{i-1}, x_i\}) \leq 1$.

Proof of Claim 13: The proof is by contradiction. By Claim 12, $y_2y_1 \in D$. Without loss of generality, we may assume that $x_{n-3}y_1$, $x_{n-2}y_1 \in D$ and y_1, x_1 are nonadjacent. It is easy to see that $d^+(y_2, \{x_1, x_2\}) = 0$, $y_1x_{n-2} \notin D$ and $y_1x_2 \notin D$ (for otherwise, if $y_1x_2 \in D$, then $x_{n-2}y_1x_2x_3\dots x_{n-3}\ x_{n-2}$ is a cycle of length $n-2$ for which $\langle\{y_2,x_1\}\rangle$ is not strong, a contradiction to Claim 12). Therefore, $A(R\to\{x_1,x_2\})=\emptyset$. Again using Claim 12, it is not difficult to check that $n\geq 6$.

Assume first that $A(R\to\{x_3,x_4,\dots,x_{n-3}\})\neq\emptyset$. Now let x_l, $l\in[3,n-3]$, be the first vertex after x_2 that $d^-(x_l,R)\geq 1$. Then $A(R\to\{x_1,x_2,\dots,x_{l-1}\})=\emptyset$ since $A(R\to\{x_1,x_2\})=\emptyset$ (in particular, $d^-(x_{l-1},R)=\emptyset$). From the minimality of l and $x_{n-2}y_1\in D$ it follows that there is a vertex $x_r\in\{x_{n-2},x_1,x_2,\dots,x_{l-2}\}$ such that $d^+(x_r,R)\geq 1$ and $A(\{x_{r+1},x_{r+2},\dots,x_{l-2}\},R)=\emptyset$ (if $x_r=x_{n-2}$, then $x_{r+1}=x_1$). This contradicts Claim 3(i) since $d^-(x_{l-1},R)=0$ and $\langle R\rangle$ is strong.

Assume next that $A(R\to\{x_3,x_4,\dots,x_{n-3}\})=\emptyset$. This together with $A(R\to\{x_1,x_2\})=\emptyset$ gives that $A(R\to\{x_1,x_2,\dots,x_{n-3}\})=\emptyset$. From this, since D is strong and $y_1x_{n-2}\notin D$, it follows that $y_2x_{n-2}\in D$. Then $x_{n-3}y_2\notin D$ and $x_{n-4}y_1\notin D$. Now using Claim 12 we obtain that $d(y_2,\{x_{n-4},x_{n-3}\})=0$. Since $d^-(x_{n-3},R)=0$ and $y_2x_{n-2}\in D$, from Claim 3(i) it follows that $d^+(x_{n-4},R)=0$. Therefore, $d(x_{n-4},R)=0$. If $A(\{x_1,x_2,\dots,x_{n-5}\}\to R)\neq\emptyset$, then there is a vertex x_r with $r\in[1,n-5]$ such that $d^+(x_r,R)\geq 1$ and $A(R,\{x_{r+1},x_{r+2},\dots,x_{n-4}\})=\emptyset$ ($n\geq 6$) which contradicts Claim 3(i), since $y_2x_{n-2}\in D$ and $d^-(x_{n-3},R)=0$. Assume therefore that $A(\{x_1,x_2,\dots,x_{n-4}\}\to R)=\emptyset$. Thus, we have that $A(\{x_1,x_2,\ \dots,x_{n-4}\},R)=\emptyset$ and $d^-(x_{n-3},R)=0$. Then $d(y_1)=4$, $d(y_2)\leq 4$ and $d(x_1)\leq 2n-6$. From this it follows that $d(y_1)+d(x_1)\leq 2n-2$ and $d(y_2)+d(x_1)\leq 2n-2$ which contradicts Lemma 3.5. This contradiction proves that $d^-(y_f,\{x_{i-1},x_i\})\leq 1$ for all $i\in[1,n-2]$ and $f\in[1,2]$. Similarly, one can show that $d^+(y_f,\{x_{i-1},x_i\})\leq 1$. Claim 13 is proved. $\square$

Claim 14: *If $x_iy_f\in D$ (respectively, $y_fx_i\in D$), then $d(y_f,\{x_{i+2}\})\neq 0$ (respectively, $d(y_f,\{x_{i-2}\})\neq 0$), where $i\in[1,n-2]$ and $f\in[1,2]$.*

Proof of Claim 14: Suppose that the claim is not true. By Claim 12, $y_2y_1\in D$. Without loss of generality, we may assume that $x_{n-2}y_1\in D$ and $d(y_1,\{x_2\})=0$, i.e., the vertices y_1 and x_2 are nonadjacent. Claim 13 implies that the vertices y_1,x_1 also are nonadjacent. Thus, $d(y_1,\{x_1,x_2\})=0$. Note that $y_2x_2\notin D$ and hence, $d^-(x_2,R)=0$. Now it is not difficult to cheek that if $n=5$, then $d(y_1)+d(x_1)\leq 8$ and $d(y_1)+d(x_2)\leq 8$, a contradiction to Lemma 3.5.

Assume, therefore, that $n\geq 6$ and consider the following cases.

Case 14.1. $A(R\to\{x_3,x_4,\dots,x_{n-3}\})\neq\emptyset$.

Then there is a vertex x_l with $l\in[3,n-3]$ such that $d^-(x_l,R)\geq 1$ and $A(R\to\{x_2,x_3,\dots,x_{l-1}\})=\emptyset$ since $d(y_1,\{x_1,x_2\})=d^-(x_2,R)=0$. We now consider the case $l=3$ and the case $l\geq 4$ separately.

Assume that $l=3$. Then $y_2x_3\in D$ or $y_1x_3\in D$.

Let $y_2x_3\in D$. Then the vertices y_2,x_2 are nonadjacent. Since the vertices y_1,x_2 are nonadjacent Claim 12 implies that $x_1y_2\notin D$ (for otherwise $x_{n-2}x_1y_2x_3\dots x_{n-4}x_{n-2}$ is a cycle of length $n-2$ which does not contain the vertices y_1,x_2 and $\langle\{y_1,x_2\}\rangle$ is not strong, a contradiction to Claim 12). This contradicts Claim 3(ii) because of $d(x_2,R)=0$ and $d^+(x_1,R)=0$.

Let now $y_1x_3\in D$ and $y_2x_3\notin D$. Then it is easy to see that $x_1y_2\notin D$ and $y_2x_2\notin D$. From this and Claim 12 implies that neither x_1 nor x_2 can be inserted into $C[x_3,x_{n-2}]$. Notice that if $x_2y_2\in D$, then $x_{n-2}x_2\notin D$, and if $y_2x_1\in D$, then $x_1x_3\notin D$. Now using Lemma 3.2, we obtain that $d(y_1)$, $d(x_1)$ and $d(x_2)\leq n-1$ since $d(y_1,\{x_1,x_2\})=0$. Therefore

$$d(y_1)+d(x_1)\leq 2n-2\quad\text{and}\quad d(y_1)+d(x_2)\leq 2n-2,$$

which contradicts Lemma 3.5 since y_1, x_1 and y_1, x_2 are two distinct pairs of nonadjacent vertices. This contradiction completes the discussion of Case 14.1 when $l = 3$.

Assume that $l \geq 4$. Let $y_g x_l \in D$, where $g \in [1, 2]$. Then, by the minimality of l, the vertices y_g, x_{l-1} are nonadjacent, $y_{3-g}x_{l-1} \notin D$ and $x_{l-2}y_{3-g} \notin D$. Hence, by Claim 12 we get that $x_{l-2}y_g \notin D$. From the minimality of l and $d^-(x_2, R) = 0$ (for $l = 4$) it follows that x_{l-2} is not adjacent to y_1 and y_2, i.e., $d(x_{l-2}, R) = 0$. This together with $d^-(x_2, R) = d^-(x_{l-1}, R) = 0$, the minimality of l and Claim 3(i) implies that

$$A(R, \{x_2, x_3, \ldots, x_{l-2}\}) = \emptyset \quad \text{and} \quad d^+(x_1, R) = 0$$

(if $d^+(x_1, R) \geq 1$, then $l \geq 5$ and there is an x_r with $r \in [1, l-3]$ such that $d^+(x_{l-1}, R) = 0$ and $A(R, C[x_{r+1}, x_{l-3}]) = \emptyset$ but this contradicts Claim 3(i)). If $d^-(x_1, R) = 0$ or $d^+(x_{l-1}, R) = 0$, then $d(x_1, R) = 0$ or $d(x_{l-1}, R) = 0$, respectively. This together with $A(R, C[x_2, x_{l-2}]) = \emptyset$ contradicts Claim 3 since $d^+(x_{n-2}, R) \geq 1$ and $d^-(x_l, R) \geq 1$. Assume, therefore, that $d^-(x_1, R) \geq 1$ and $d^+(x_{l-1}, R) \geq 1$. It follows that $y_2 x_1 \in D$ since $y_1 x_1 \notin D$.

Assume first that $y_g = y_2$. Then $x_{l-1}y_1 \in D$. Since $y_1 x_{l-1} \notin D$, $x_1 y_2 \notin D$ and

$$d(y_1, C[x_1, x_{l-2}]) = d(y_2, C[x_2, x_{l-1}]) = 0$$

using Lemma 3.2(ii) we obtain that

$$d(y_1) = d(y_1, \{y_2\}) + d(y_1, C[x_{l-1}, x_{n-2}]) \leq n - l + 2 \quad \text{and}$$

$$d(y_2) = d(y_2, \{y_1\}) + d(y_2, C[x_l, x_1]) \leq n - l + 2. \tag{17}$$

Now we extend the path $P_0 := C[x_l, x_{n-2}]$ with the vertices $x_1, x_2, \ldots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_d \in \{x_1, x_2, \ldots, x_{l-1}\}$, $d \in [2, l-1]$, are not on the extended path P_e since otherwise P_{e-1} or P_e together with the arcs $x_{n-2}y_1, y_1y_2$ and y_2x_l forms a cycle of length $n-1$. Therefore, by Lemma 3.2, we have that $d(z_i, C) \leq n + d - 3$. If there is a $z_i \notin \{x_1, x_{l-1}\}$, then $d(z_i) \leq n + d - 3$ and by (17),

$$d(z_i) + d(y_1) \leq 2n - 2 \quad \text{and} \quad d(z_i) + d(y_2) \leq 2n - 2,$$

which contradicts Lemma 3.5 since z_i is not adjacent to y_1 and y_2. Therefore, assume that $\{z_1, z_2\} = \{x_1, x_{l-1}\}$ ($d = 2$). Then P_e ($e = l - 3 \geq 1$) is an (x_l, x_{n-2})-path with vertex set $V(C) - \{x_1, x_{l-1}\}$. Thus, we have that $y_2 P_e y_1 y_2$ is a cycle of length $n-2$. Therefore, by Claim 12, $x_1 x_{l-1} \in D$, and hence, $x_1 x_{l-1} P_{e-1} y_1 y_2 x_1$ is a cycle of length $n-1$, which contradicts the initial supposition that D contains no cycle of length $n-1$.

Assume second that $y_g = y_1$. Then by the above observation we conclude that $x_{l-1}y_2 \in D$ and $d(y_1, C[x_1, x_{l-1}]) = 0$. Using Lemma 3.2, we obtain that for this case (17) also holds, since $x_1 y_2 \notin D$ and $y_2 x_{l-1} \notin D$. Again we extend the path $C[x_l, x_{n-2}]$ with vertices $x_1, x_2, \ldots, x_{l-1}$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_d \in \{x_1, x_2, \ldots, x_{l-1}\}$, $d \in [1, l-1]$, are not on the extended path P_e. Similar to the first case when $y_g = y_2$, we will obtain that $z_i \notin \{x_2, x_3, \ldots, x_{l-2}\}$ (i.e., $z_i = x_1$ or $z_i = x_{l-1}$) and $d(z_i) \leq n + d - 2$. Notice that $C' := y_1 P_e y_1$ is a cycle of length $n - d - 1$ with vertex set $V(C) \cup \{y_1\} - \{z_1, z_d\}$. From Claim 12 it follows that $d = 2$, i.e., $\{z_1, z_d\} = \{x_1, x_{l-1}\}$ (since $x_1 y_2 \notin D$ and $y x_{l-1} \notin D$). Now from $l \geq 4$, $d = 2$, (17) and $d(z_i) \leq n + d - 2$ we obtain that

$$d(y_1) + d(x_1) \leq 2n - 2 \quad \text{and} \quad d(y_1) + d(x_{l-1}) \leq 2n - 2,$$

which contradicts Lemma 3.5, since y_1, x_1 and y_1, x_{l-1} are two distinct pairs of nonadjacent vertices.

Case 14.2. $A(R \to \{x_3, x_4, \dots, x_{n-3}\}) = \emptyset$.

Then $A(R \to \{x_{n-2}, x_1\}) \neq \emptyset$ since $d^-(x_2, R) = 0$ and D is strong, and y_1, x_{n-3} are nonadjacent (by Claim 13). For this case we distinguish three subcases.

Subcase 14.2.1. $y_2x_{n-2} \in D$.

Then, using Claim 13, it is easy to see that x_{n-3}, y_2 are nonadjacent. Therefore, $d(x_{n-3}, R) = 0$. This together with $y_2x_{n-2} \in D$ and Claim 3 implies that $A(\{x_1, x_2, \dots, x_{n-3}\} \to R) = \emptyset$. Therefore, $d(R, \{x_2, x_3, \dots, x_{n-3}\}) = \emptyset$ and $d(y_1)$, $d(y_2) \leq 4$ (since $y_2x_1 \notin D$ by Claim 13) and $d(x_{n-3}) \leq 2n-6$. From this it follows that $d(y_1) + d(x_{n-3}) \leq 2n-2$ and $d(y_2) + d(x_{n-3}) \leq 2n-2$, which contradicts Lemma 3.5.

Subcase 14.2.2. $y_2x_{n-2} \notin D$ *and* $y_2x_1 \in D$.

Then using Claim 13 it is easy to see that y_2 and x_{n-2} are nonadjacent.

Let $x_{n-3}y_2 \in D$. Then $y_1x_{n-2} \in D$ (by Claim 12). Using Claims 12 and 13 we obtain that x_{n-4} is not adjacent to y_1 and y_2. Since $d^-(x_{n-3}, R) = 0$ and $y_1x_{n-2} \in D$, from Claim 3(i) it follows that $A(\{x_1, x_2, \dots, x_{n-4}\} \to R) = \emptyset$ and $A(R, C[x_2, x_{n-4}]) = \emptyset$. Therefore and $d(y_1) = d(y_2) = 4$ and $d(x_2) \leq 2n-6$. From these it follows that

$$d(y_1) + d(x_2) \leq 2n-2 \quad \text{and} \quad d(y_2) + d(x_2) \leq 2n-2,$$

which contradicts Lemma 3.5 since x_2, y_1 and x_2, y_2 are two distinct pairs of nonadjacent vertices.

Let now $x_{n-3}y_2 \notin D$. Then y_2, x_{n-3} are nonadjacent and hence, $d(x_{n-3}, R) = 0$. Now, since $y_1x_{n-2} \in D$ or $d^-(x_{n-2}, R) = 0$ and $y_2x_1 \in D$, from Claim 3 it follows that $A(\{x_2, x_3, \dots, x_{n-3}\} \to R) = \emptyset$. Therefore

$$d(y_1, C[x_1, x_{n-3}]) = d(y_2, C[x_2, x_{n-2}]) = 0,$$

$d(y_1) \leq 4$, $d(y_2) \leq 4$ and $d(x_2) \leq 2n-6$. This contradicts Lemma 3.5 since x_2, y_1 and x_2, y_2 are two distinct pairs of nonadjacent vertices.

Subcase 14.2.3. $y_2x_{n-2} \notin D$ *and* $y_2x_1 \notin D$.

Then $y_1x_{n-2} \in D$ (since D is strong), the vertex y_1 is not adjacent to the vertices x_{n-3}, x_{n-4} and $x_{n-4}y_2 \notin D$, i.e., the vertices y_2, x_{n-4} also are nonadjacent. Using Claim 3, we can assume that $A(C[x_1, x_{n-4}] \to R) = \emptyset$. Therefore, $d(y_1) = 4$, $d(y_2) \leq 3$ and $d(x_1) \leq 2n-6$. This contradicts Lemma 3.5 since x_1 is not adjacent to y_1 and y_2. This completes the proof of Claim 14. □

Claim 15: *If* $x_iy_f \in D$ *and the vertices* y_f, x_{i+1} *are nonadjacent, then the vertices* x_{i+1}, y_{3-f} *are adjacent, where* $i \in [1, n-2]$ *and* $f \in [1, 2]$.

Proof of Claim 15: Without loss of generality, we may assume that $x_i = x_{n-2}$ (i.e., $x_{i+1} = x_1$) and $y_f = y_1$. Suppose, on the contrary, that x_1, y_2 are nonadjacent. From Claims 12 and 14 it follows that $y_1x_2 \notin D$ and $x_2y_1 \in D$. Therefore, $A(R \to \{x_1, x_2\}) = \emptyset$. If $n = 5$, then $x_2y_1, x_3y_1 \in D$ which contradicts Claim 13. Assume, therefore, that $n \geq 6$. As D is strong, there is a vertex x_l with $l \in [3, n-2]$ such that $d^-(x_l, R) \geq 1$ (say $y_gx_l \in D$) and $A(R \to C[x_1, x_{l-1}]) = \emptyset$. Then the vertices x_{l-1}, y_g are nonadjacent and $d(x_{l-2}, R) = 0$ (by $x_{l-2}y_{3-g} \notin D$ and by Claim 12). Now, since $x_{n-2}y_1$ and $x_2y_1 \in D$, there exists a vertex $x_r \in C[x_{n-2}, x_{l-3}]$ (if $l = 3$, then $x_{n-2} = x_{l-3}$) such that $d^+(x_r, R) \geq 1$ and

$A(R, C[x_{r+1}, x_{l-2}]) = \emptyset$. This contradicts Claim 3 since $d^-(x_{l-1}, R) = 0$ and $d^-(x_l, R) \geq 1$. Claim 15 is proved.

Claim 16: *If $x_iy_j \in D$, where $i \in [1, n-2]$ and $j \in [1, 2]$, then $y_jx_{i+2} \in D$.*

Proof of Claim 16: Without loss of generality, we may assume that $x_i = x_{n-2}$ and $y_j = y_1$. Suppose that the claim is not true, that is $x_{n-2}y_1 \in D$ and $y_1x_2 \notin D$. Then, by Claims 13 and 14, the vertices y_1, x_1 are nonadjacent, $x_2y_1 \in D$ (hence, $n \geq 6$) and y_1, x_3 are also nonadjacent. From this, by Claim 15 we obtain that the vertex y_2 is adjacent to the vertices x_1 and x_3. Therefore either $y_2x_3 \in D$ or $x_3y_2 \in D$.

Case 16.1. $y_2x_3 \in D$.

Then x_2, y_2 are nonadjacent (by Claim 13), $x_2x_1 \in D$ and $x_1y_2 \notin D$ by Claim 12 (for otherwise D would have a cycle C' of length $n-2$ for which $\langle V(D) - V(C')\rangle$ is not strong). Notice that $y_2x_1 \in D$ (by Claim 15). Since neither y_1 nor y_2 can be inserted into C, $y_1x_2 \notin D$ and y_1, x_1 are nonadjacent (respectively, $x_1y_2 \notin D$ and y_2, x_2 are nonadjacent) using Lemma 3.2(ii), we obtain that

$$d(y_1) \leq n-1 \quad \text{and} \quad d(y_2) \leq n-1. \tag{18}$$

It is not difficult to see that $x_{n-2}x_2 \notin D$ and $x_1x_3 \notin D$ (for otherwise D contains a cycle of length $n-1$). Therefore, since neither x_1 nor x_2 cannot be inserted into $C[x_3, x_{n-2}]$ (otherwise we obtain a cycle of length $n-1$), again using Lemma 3.2(ii), we obtain

$$d(x_1) \leq n-1 \quad \text{and} \quad d(x_2) \leq n-1. \tag{19}$$

It is easy to check that $n \geq 7$.

Remark: *Observe that from (18), (19) and Lemma 3.5 it follows that if $x_i \neq x_1$ and y_1, x_i are nonadjacent or $x_i \neq x_2$ and x_i, y_2 are nonadjacent, then $d(x_i) \geq n$.*

Assume first that $d^+(y_1, C[x_4, x_{n-2}]) \geq 1$. Let x_l, $l \in [4, n-2]$, be the first vertex after x_3 that $y_1x_l \in D$. Then the vertices y_1 and x_{l-1} are nonadjacent. Therefore, y_1 and x_{l-2} are adjacent (by Claim 14) and hence, $x_{l-2}y_1 \in D$ because of $x_2y_1 \in D$ and minimality of l ($l-1 \neq 4$ by Claim 14, since $x_2y_1 \in D$). Since x_{l-1} cannot be inserted into $C[x_l, x_{l-2}]$, using Lemma 3.2 and the above Remark, we obtain that $d(x_{l-1}) = n$ and hence, $d(y_1) = n-1$ (by Lemma 3.5). This together with $d(y_1, \{x_1, x_2, x_3, y_2\}) = 3$ implies that $d(y_1, C[x_4, x_{n-2}]) = n-4$. Again using Lemma 3.2, we obtain that $y_1x_4 \in D$ (since $|C[x_4, x_{n-2}]| = n-5$). Thus, $y_1C[x_4, x_2]y_1$ is a cycle of length $n-2$. Therefore, $x_3y_2 \in D$ (by Claim 12), $y_1x_5 \notin D$ and the vertices y_2, x_4 are nonadjacent (by Claim 13). From $y_1x_5 \notin D$ (by Lemma 3.2) we obtain that $d(y_1, C[x_5, x_{n-2}]) \leq n-6$. Therefore $x_4y_1 \in D$ and $d(y_1, C[x_5, x_{n-2}]) = n-6$. Now it is easy to see that y_1, x_5 are nonadjacent (by Claim 13) and y_2, x_5 are adjacent (by Claim 14). Therefore, $d(y_1, C[x_6, x_{n-2}]) = n-6$ and $y_1x_6 \in D$ (by Lemma 3.2), $y_2x_5, x_5y_2 \in D$ (by Claim 12), $y_1x_7 \notin D$ (by Claim 13). One readily sees that, by continuing the above procedure, we eventually obtain that n is even and

$$N^-(y_1) = \{y_2, x_2, x_4, x_6, \ldots, x_{n-2}\}, \quad N^+(y_1) = \{y_2, x_4, x_6, \ldots, x_{n-2}\},$$

$$N^-(y_2) = \{y_1, x_3, x_5, \ldots, x_{n-3}\}, \quad N^+(y_2) = \{y_1, x_1, x_3, x_5, \ldots, x_{n-3}\}.$$

From Claim 12 it follows that $x_ix_{i-1} \in D$ for all $i \in [4, n-2]$ and $x_2x_1 \in D$. It is easy to see that $x_1x_3 \notin D$ and $x_3x_5 \notin D$. Therefore, since x_3 cannot be inserted into $C[x_5, x_1]$, by Lemma 3.2, we have $d(x_3, C[x_5, x_1]) \leq n-6$. This together with $d(x_3) \geq n$ (by Remark) implies that $d(x_3, \{x_2, x_4, y_2\}) = 6$. In particular, $x_3x_2 \in D$. Now we consider the vertex x_{n-2}. Note that $d(x_{n-2}) \geq n$ (by Remark), $x_{n-2}x_2 \notin D$ and $x_{n-4}x_{n-2} \notin D$. From this it is not difficult to see that $d(x_{n-2}, C[x_2, x_{n-4}]) \leq n-6$ and $x_1x_{n-2} \in D$. It follows that $x_{n-2}x_{n-3}\ldots x_4x_3y_2x_1x_{n-2}$ is a cycle of length $n-2$, which does not contain the vertices y_1 and x_2. This contradicts Claim 12, since $y_1x_2 \notin D$ (by our supposition), i.e., $\langle\{y_1, x_2\}\rangle$ is not strong.

Assume next that $d^+(y_1, C[x_4, x_{n-2}]) = 0$. Then from Claims 13 and 14 it follows that

$$N^-(y_1) = \{y_2, x_2, x_4, \ldots, x_{n-2}\} \quad \text{and} \quad N^+(y_1) = \{y_2\}. \tag{20}$$

By Claim 15 we have that the vertex y_2 is adjacent to each vertex $x_i \in \{x_1, x_3, \ldots, x_{n-3}\}$. It is easy to see that $x_{n-3}y_2 \notin D$ and hence, $y_2x_{n-3} \in D$ (for otherwise if $x_{n-3}y_2 \in D$, then $y_2C[x_1, x_{n-3}]y_2$ is a cycle of length $n-2$, but $\langle\{x_{n-2}, y_1\}\rangle$ is not strong, a contradiction to Claim 12). By an argument similar to that in the proof of (20) we deduce that

$$N^+(y_2) = \{y_1, x_1, x_3, \ldots, x_{n-3}\} \quad \text{and} \quad N^-(y_2) = \{y_1\}.$$

Thus we have that $y_1y_2C[x_5, x_2]y_1$ is a cycle of length $n-2$ and x_3 cannot be inserted into $C[x_5, x_2]$. Therefore, by Lemma 3.2(ii), $d(x_3, C[x_5, x_2]) \leq n-4$ since $x_3x_5 \notin D$. This together with $d(x_3, \{x_4, y_1, y_2\}) \leq 3$ implies that $d(x_3) \leq n-1$ which contradicts the above Remark that $d(x_3) \geq n$.

Case 16.2. $y_2x_3 \notin D$.

Then, as noted above, $x_3y_2 \in D$. Therefore $d(y_2, \{x_2, x_4\}) = 0$ (by Claim 13 and $y_2x_2 \notin D$), $y_1x_4 \notin D$ (by Claim 12), $x_4y_1 \in D$ (by Claim 15), the vertices x_5, y_1 are nonadjacent and the vertices y_2, x_5 are adjacent (by Claim 15). Since $x_3y_2 \in D$, $y_1x_4 \notin D$ and y_2, x_5 are adjacent, from Claim 12 it follows that $y_2x_5 \notin D$ and $x_5y_2 \in D$. For the same reason, we deduce that

$$N^-(y_1) = \{y_2, x_2, x_4, \ldots, x_{n-2}\} \quad N^-(y_2) = \{y_1, x_1, x_3, \ldots, x_{n-3}\} \quad \text{and} \quad A(R \to V(C)) = \emptyset,$$

which contradicts that D is strong. This contradiction completes the proof of Claim 16. □

We will now complete the proof of Theorem by showing that D is isomorphic to $K^*_{n/2,n/2}$. Without loss of generality, we assume that $x_{n-2}y_1 \in D$. Then using Claims 12, 13, 14 and 16 we conclude that y_1, x_1 are nonadjacent (Claim 13), $y_1x_2 \in D$ (Claim 16), $x_1y_2, y_2x_1 \in D$ (Claim 12), x_2, y_2 also are nonadjacent (Claim 13), $y_2x_3 \in D$ (Claim 16) and $x_2y_1 \in D$ (Claim 12). By continuing this procedure, we eventually obtain that n is even and

$$N^+(y_1) = N^-(y_1) = \{y_2, x_2, x_4, \ldots, x_{n-2}\} \quad \text{and} \quad N^+(y_2) = N^-(y_2) = \{y_1, x_1, x_3, \ldots, x_{n-3}\}.$$

If $x_ix_j \in D$ for some $x_i, x_j \in \{x_1, x_3, \ldots, x_{n-3}\}$, then clearly $|C[x_i, x_j]| \geq 5$ and $x_ix_jx_{j+1}\ldots x_{i-1}y_1x_{i+1}\ldots x_{j-2}y_2x_i$ is a cycle of length $n-1$, contrary to our assumption. Therefore, $\{y_1, x_1, x_3, \ldots, x_{n-3}\}$ is an independent set of vertices. For the same reason $\{y_2, x_2, x_4, \ldots, x_{n-2}\}$ also is an independent set of vertices. Now from the condition A_0 it follows that D is isomorphic to $K^*_{n/2,n/2}$. This completes the proof of Theorem 1.10. □

5. Concluding Remarks

A Hamiltonian bypass in a digraph is a subdigraph obtained from a Hamiltonian cycle of D by reversing one arc.

Using Theorem 1.10, the first author has proved that if a strong digraph D of order $n \geq 4$ satisfies the condition A_0, then D contains a Hamiltonian bypass or D is isomorphic to one tournament of order 5.

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Կողմնորոշված համիլտոնյան գրաֆների նախահամիլտոնյան ցիկլերի մասին

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Ամփոփում

Կողմնորոշված գրաֆի կողմնորոշված ցիկլը, որն անցնում է նրա բոլոր գագաթներով, բացի մեկից, կոչվում է նախահամիլտոնյան ցիկլ: Ներկա աշխատանքում ապացուցված է, որ եթե կողմնորոշված գրաֆը բավարարում է Մանոուսակիսի համիլտնյանության բավարար պայմանին (J. of Graph Theory 16(1) (1992) 51-59), ապա այն պարունակում է նախահամիլտոնյան ցիկլ, բացի այն դեպքից, երբ այդ գրաֆը իզոմորֆ է երկմաս հավասարակշռված կողմնորոշված լրիվ գրաֆին:

О предгамильтоновых контурах в гамильтоновых ориентированных графах

С. Дарбинян и И. Карапетян

Аннотация

Ориентированный контур, который содержит все вершины ориентированного графа (орграфа), называется предгамильтоновым контуром. В работе доказано, что любой орграф, который удовлетворяет достаточному условию гамильтоновости орграфов Маноусакиса (J. of Graph Theory 16(1) (1992) 51-59), содержит предгамильтоновый контур или является двудольным балансированным полным орграфом.