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Gossiping Properties of the Modified Knödel Graphs

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Abstract

In this paper we consider the gossiping process implemented on several modifications of Knödel graphs. We show the ability of Knödel graphs to remain good network topology for gossiping even in case of cyclic permutation of its edge weights. The results shown in this paper could help us to construct edge-disjoint paths between any pairs of vertices of the Knödel graph.

Keywords: Graphs, Networks, Telephone problem, Gossip problem, Knödel graphs.

1. Introduction

Gossiping is one of the basic problems of information dissemination in communication networks. The gossip problem (also known as a telephone problem) is attributed to A. Boyd (see e.g., [4] for review), although to the best knowledge of the reviewers, it was first formulated by R. Chesters and S. Silverman (Univ. of Witwatersrand, unpublished, 1970). Consider a set of n persons (nodes) each of which initially knows some unique piece of information that is unknown to the others, and they can make a sequence of telephone calls to spread the information. During a call between the given two nodes, they exchange the whole information known to them at that moment. The problem is to find a sequence of calls with a minimum length (minimal gossip scheme), by which all the nodes will know all pieces of the information (complete gossiping). It has been shown in numerous works [1]–[4] that the minimal number of calls is $2n-4$ when $n \geq 4$ and 1, 3 for $n = 2, 3$, respectively. Since then many variations of gossip problem have been introduced and investigated (see e.g., [5]–[10], [13]–[17]).

Another variant of Gossip problem can be formulated by considering the minimum amount of time required to complete gossiping among n persons, where the calls between the non-overlapping pairs of nodes can take place simultaneously and each call requires one unit of time ([11], [12],[18]).

Obviously, the gossip problem can be easily modeled as a graph, the vertices of which represent people in gossip scheme and edges represent calls between them (each of them has weight which represents the moment when communication took place). So the graph is called a complete gossip graph if there are ascending paths between all the pairs of vertices of this graph. A path between two vertices is an ascending path if each next edge has higher weight than the previous one.

Knödel graphs are one of the 3 well-known network topologies for gossiping/broadcasting (alongside with hypercube and recursive circulant graphs).

Definition 1: *The Knödel graph on* $n \geq 2$ *vertices (n even) and of maximum degree* $\Delta \geq 1$ *is denoted by* $W_{\Delta,n}$. *The vertices of* $W_{\Delta,n}$ *are the pairs* (i,j) *with* $i = 1,2$ *and* $0 \leq j \leq n/2-1$. *For every* j, $0 \leq j \leq n/2-1$ *and* $l = 1,\ldots,\Delta$, *there is an edge with the label (weight)* l *between the vertex* $(1,j)$ *and* $(2,(j+2^{l-1}-1) \bmod n/2)$. *The edges with the given label* l *are said to be in dimension* l.

Note that $W_{1,n}$ consists of $n/2$ disconnected edges. For $\Delta \geq 2$, $W_{\Delta,n}$ is connected.

In this paper we give some new observations, mostly in terms of gossiping, concerning a slight modification of Knödel graphs, obtained by a cyclic permutation of the dimensions of edges. For the graph under consideration, the modification looks as follows: dimension of the edge connecting the vertices $(1,j)$ and $(2,(j+2^{l-1}-1) \bmod n/2)$, is replaced by $(l+p) \bmod \lceil \log_2 n \rceil$ where $p = 1,\ldots,\lceil\Delta\rceil$. We first consider the case of $n = 2^k-2$ and show the ability of Knödel graphs to preserve gossiping properties for the operation described. Then we try to generalize the above statement about Knödel graphs for all generic $n \neq 2^k$. At the end we show the inability of these graphs to preserve gossiping properties in the case of $n = 2^k$, except for one specific case.

2. Modified Knödel Graphs With $n = 2^k - 2$ Vertices

In this section we are going to consider the modified Knödel graphs with the number of vertices equal to $2^k - 2$, where k is any integer with $k \geq 3$. The definition of modified Knödel graph is as follows:

Definition 2: *The modified Knödel graph on* $n \geq 2$ *vertices (n even) and of maximum degree* $\Delta \geq 1$ *is denoted by* $M_{\Delta,n}(p)$. *The vertices of* $M_{\Delta,n}(p)$ *are the pairs* (i,j) *with* $i = 1,2$ *and* $0 \leq j \leq n/2-1$, *respectively. For every* j, $0 \leq j \leq n/2-1$ *and* $l = 1,\ldots,\Delta$, *there is an edge with the label* $(l+p) \bmod \lceil \log_2 n \rceil$ *between the vertices* $(1,j)$ *and* $(2,(j+2^{l-1}-1) \bmod n/2)$ *where* p *is a fixed integer for that graph with possible values* $p = 1,\ldots,\Delta$. *The edges with the given label* $l+p$ *are said to be of dimension* $l+p$.

According to the above definition, there exist $\Delta - 1$ modifications for each $W_{\Delta,n}$. In this paper we consider only the case, when $\Delta = \lceil \log_2 n \rceil$, and this value is assumed for all references of Δ onwards.

Definition 3: *Consider two graphs* $G_1 = (V,E_1)$ *and* $G_2 = (V,E_2)$ *with the same set of vertices* V *and labeled edge sets* E_1 *and* E_2, *respectively. The edge sum of these graphs is a graph* $G_1 + G_2 = G = (V,E)$ *with* $E = E_1 \cup E_2$, *whose edges* $e \in E$ *are labeled by the following rules:*

$$t_G(e) = \begin{cases} t_{G_1}(e), & \text{if } e \in E_1, \\ t_{G_2}(e) + \max\limits_{e' \in E_1} t_{G_1}(e'), & \text{if } e \in E_2. \end{cases} \tag{1}$$

where $t_G(e)$ *is tha label of the edge* e *in the resulting graph* G.

It is known that if $G = W_{\Delta,n} + W_{1,n}$, then G is a complete gossip graph.

Let us first consider the graph $G' = M_{\Delta,n}(\Delta) + M_{1,n}(\Delta)$.

Theorem 1: *The graph* G' *is a complete gossip graph.*

Proof: Since this graph is symmetric, it is enough to show that there exists an ascending path from any other vertex to the vertex $(0, 1)$. Let us consider the subgraph of $G\prime$, $SUB_{G\prime}$ that includes the vertices (i, j), where $i = 1, 2$ and $1 \leq j \leq \Delta$. Let us particularly focus on the vertices, for which $i = 0$ and $1 \leq j \leq \Delta$, and consider the call in dimension 1. There is an edge in dimension 1 between these and $(1, j')$ vertices, where $2^\Delta \leq j' \leq n/2$. The same is true about the vertices $(1, j)$ and $(0, j')$, where $1 \leq j \leq 2^\Delta$ and $2^\Delta < j' \leq n/2$. It is obvious that these edges connect all the remaining $2^k - 2 - 2^\Delta = 2^{\Delta+1} - 2^\Delta - 2 = 2^\Delta - 2$ vertices to the $SUB_{G'}$. On the other hand, we have that $SUB_{G'}$ is a tree rooted at the vertex $(0, 1)$. Thus, there exist ascending paths between every vertex of $SUB_{G'}$ and the vertex $(0, 1)$. Considering the fact that each edge in the $SUB_{G'}$ has dimension greater than 1, it follows that there exists an ascending path between any other vertex and the vertex $(0, 1)$. The two facts that the choice of the vertex $(0, 1)$ was arbitrary, also the modified Knödel graph being symmetric by construction, point out to complete gossiping of G'. ■

Corollary 2: *Since the graph G' is a gossip graph, we can claim that $M_{\Delta,n}(p)$, where $0 \leq p \leq \Delta$, is also a gossip graph.*

Proof. It is an easy exercise to verify that G' is isomorphic to $W_{\Delta,n}$ (for detailed proof refer to [23]). Hence, we can repeat this procedure, and each time we will obtain a new $M_{\Delta,n}(p)$, $p = \Delta - 1, \Delta - 2, ..., 1$, that is isomorphic to initial $W_{\Delta,n}$. Thus, there is no need to add $W_{1,n}$ in order to obtain a complete gossip graph. ■

3. Modified Knödel Graphs With $n = 2^k$ Vertices

In this section we consider modified Knödel graphs with 2^k vertices. Knödel graphs with 2^k vertices are different in terms of gossiping since $W_{\Delta,n}$ is actually a gossip graph. Hence, in this case there is no need to add $W_{1,n}$ to get a complete gossip graph.

To start with, let us consider $M_{\Delta,n}(2)$.

Theorem 3: *The graph $M_{\Delta,n}(2)$ is a complete gossip graph.*

Proof: Let's start our proof by doing the following transformation on the vertex set of this graph V. The elements of this set are the pairs (i, j). In case of $i = 1$, if j is odd, then $j_{new} = \lceil j/2 \rceil$ and $j_{new} = n/2 + j/2$, otherwise. If $i = 2$, the rules are as follows: $j_{new} = j/2$ in case j is even, and $j_{new} = n/2 + \lfloor j/2 \rfloor$ in case of odd j.

We obtain two components that are Knödel graphs with the number of vertices equal to $n/2$, and these components are connected by the edges of label Δ. Therefore, if we simply remove these edges, then there will be two disjoint components which are Knödel graphs themselves. And there is an one-to-one call mapping between these two components with the calls labeled by $\log_2 n$ (highest labeled edges). Obviously, this graph is a complete gossip graph. ■

Theorem 4: *The graph $M_{\Delta,n}(p)$, where $p \neq 2$, is not a gossip graph.*

Proof: To start with the proof, let us recall our method for modifying gossip graphs, firstly introduced in [22]. The operations permute higher and permute lower were introduced to modify the topology of the graph without changing its main gossiping properties (number of edges, number of rounds, etc.).

Let us denote the subset of the edge set of graph M with the weight $w_{fix} = log_2 n - i + 2$ by E', and its elements by e'_l, where $l = 1, 2, \ldots n/2$. The operation $A^-_{e'_l}$ (see [22]) should apply the "permute lower" operation on all the edges of E' resulting in a completly new graph. There would obviously be duplicate (double) edges in this graph, since the edges with weights $w_{fix} - 1$ and $w_{fix} + 1$ would connect the same vertices. Hence, taking into consideration the fact that the gossiping time, i.e., the number of rounds required to complete gossiping in $W_{\Delta,n}$ was $log_2 n$ (which is the most optimal value of this property), the graph $M_{\Delta,n}(p)$ is proved to be a non-gossip graph. ■

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Մոդիֆիկացված Knödel գրաֆների gossiping (բամբասանքային) հատկությունները

Վ. Հովնանյան

Ամփոփում

Այս հոդվածում մենք դիտարկում ենք Knödel գրաֆների որոշ մոդիֆիկացիաների վրա իրականացվող gossiping պրոցեսը: Ցույց է տրվում Knödel գրաֆի gossiping-ի համար լավ ցանցային տոպոլոգիա հանդիսանալու կարողությունը նույնիսկ դրա կողերի կշիռների ցիկլիկ վերադասավորման պարագայում: Հոդվածում ցույց տրված արդյունքները կարող են օգտակար լինել Knödel գրաֆի կամայական երկու գագաթների միջև կողանկախ ճանապարհների կառուցման համար:

Госсип свойства модифицированных Кнедел графов

В. Овнанян

Аннотация

В этой статье мы рассматриваем процесс госсипа реализованный на некоторых модификациях Кнедел графов. Показана способность Кнедел графов оставаться хорошей сетевой топологией для госсипа даже в случае циклической перестановки весов ребер. Результаты, представленные в данной работе способствуют построению реберно-непересекающихся путей между любыми парами вершин Кнедел графов.