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Linear Orderings of Tridimensional Grids

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Abstract

The minimal linear arrangement problem (MinLA) is defined as follows: given a graph G, find a linear ordering (layout) φ for the vertices of G on a line such that the sum of the edge lengths is minimized over all orderings. Edge length for an edge (x, y) is defined as $|\varphi(x) - \varphi(y)|$. In this paper we describe the class of minimal orderings of the special case of tridimensional grids – Cartesian product of three simple paths, when one of them consists of two vertices.

Keywords: Linear ordering, Minimal Linear Arrangement Problem, Grids, Wirelength.

1. Introduction

Given a graph G=(X,U), a layout φ is a one-to-one mapping φ : X →{1,…,|X|}. For a given graph G=(X,U) and a layout φ, we define

$$E_{\varphi}(G) = \sum_{(x,y)\in U} |\varphi(x) - \varphi(y)|,$$

as a wirelength of φ. We define also wirelength of G as $E(G) = \min_{\varphi} E_{\varphi}(G)$, where φ ranges over all layouts of G, and a layout φ_0 is called minimal if $E_{\varphi_0}(G)$= $E(G)$. Let's denote by ϕ_G^E the class of minimal layouts of G.

Let $X', X'' \subset X$ be nonempty disjoint sets, $k \in \overline{1,N}$ and φ be some layout of G. Let's denote:

$$X_{\varphi}^{k} = \{\varphi^{-1}(1), \varphi^{-1}(2), \dots, \varphi^{-1}(k), \}$$

$$\omega(X', X'') = |\{(x,y) \in U \ / \ x \in X'; y \in X''\}|$$

$$\delta_{\varphi}(X') = \frac{1}{|X'|}(|\{(x,y) \in U \ / \ x \in X'; y \notin X'; \varphi(x) < \varphi(y)\}| - |\{(x,y) \in U \ / \ x \in X'; y \notin X'; \varphi(x) > \varphi(y)\}|)$$

Definition: *We say that a set X' ($X' \subset X$) is compact with respect to layout φ, if*

$$\max_{x\in X'}\varphi(x)-\min_{x\in X'}\varphi(x)=|X'|-1.$$

Definition: *We say that a set X' ($X' \subset X$) directly goes behind the set X'' ($X'' \subset X$) (this is denoted by $X' \overset{\varphi}{\leftarrow} X''$), if X', X'' are compact and $\max_{x\in X'}\varphi(x)=\min_{x\in X''}\varphi(x)+1$.*

Definition: *We say that the sets X', X'' are independent of one another, if $\omega(X', X'')=0$.*

In the present paper the following lemma from [1] will play an essential role.

Lemma: *If $\omega(X', X'')=0$, $X' \overset{\varphi}{\leftarrow} X''$ and φ is a minimal layout, then*

$$.\ \delta_\varphi(X')\le\delta_\varphi(X'')$$

Let φ be some layout of G=(X,U) and G' be an induced subgraph with vertex set $X' \subset X$. Let the vertices of X' have the following numbers at the layout φ:

$$a_1<a_2<\cdots<a_{|X'|}.$$

Consider the following layout φ':

$$\varphi'\left(\varphi^{-1}(a_i)\right)=i\quad\left(i=\overline{1,|X'|}\right).$$

Definition: *We say that a subgraph G' is ordered minimally at φ, if φ' is a minimal layout for G'.*

Consider the graph $P^{2,m,n}$ with the vertex set $\Pi^{2,m,n}=\left\{x_{i,j,k}/\ i=\overline{1,2}; j=\overline{1,m}; k=\overline{1,n}\right\}$ and the edge set U, where $\left(x_{i,j,k}, x_{i',j',k'}\right)\in U$ if and only if $|i-i'|+|j-j'|+|k-k'|=1$.

Let's denote

$$\Pi_{i_1,j_1,k_1}^{i_2,j_2,k_2}=\left\{x_{i,j,k}\ /\ i_1\le i\le i_2;\ j_1\le j\le j_2;\ k_1\le k\le k_2\right\},$$

$$\Omega_0=\left\{x_{1,1,1}, x_{1,m,1}, x_{1,1,n}, x_{1,m,n}, x_{2,1,1}, x_{2,m,1}, x_{2,1,n}, x_{2,m,n}\right\}$$

where $1\le i_1\le i_2\le 2;\ 1\le j_1\le j_2\le m;\ 1\le k_1\le k_2\le n.$

Definition: *We say that the set $X'\subset\Pi^{2,m,n}$ is concise with respect to $x_{1,1,1}$, if for every $x_{i,j,k}\in X'$ we have $\Pi_{1,1,1}^{i,j,k}\subseteq X'$.*

Definition: *We say that a layout φ is concise with respect to $x_{1,1,1}$, if for every $k\in\overline{1,2mn}$ the set X_φ^k is concise with respect to $x_{1,1,1}$.*

Similarly one can define conciseness of sets and layouts with respect to other vertices from Ω_0.

The following statements are valid.

1. If $\varphi\in\phi_{P2,m,n}^{E}$, then for every $k\in\overline{1,2mn}$ the set X_φ^k is concise with respect to at least one vertex from Ω_0.
2. For each vertex from Ω_0, there is a minimal, concise with respect to its layout.

We will leave out the proofs of the above statements as they are very similar to analogous statements from [1]

The following theorem is a main result of this paper.

Theorem: *Let φ be concise with respect to $x_{1,1,1}$. Then φ is minimal if and only if for each i,j ($i \in \overline{1,n}; j \in \overline{1,m}$) $x_{1,i,j} \overset{\varphi}{\leftarrow} x_{2,i,j}$ and the subgraphs induced by the sets $\Pi_{1,1,1}^{1,m,n}, \Pi_{2,1,1}^{2,m,n}$ are ordered minimally at φ.*

Proof: Only taking into consideration conciseness of φ with respect to $x_{1,1,1}$, the set $\Pi^{2,m,n}$ is divided into subsets Π_i (regarding $\delta_\varphi(x)$):

$\Pi_0 = \{x_{1,1,1}\}$

$\Pi_1 = \Pi_{1,2,1}^{1,m-1,1} \cup \Pi_{1,1,2}^{1,1,n-1}$;

$\Pi_2 = \Pi_{1,2,2}^{1,m-1,n-1} \cup \{x_{1,m,1}, x_{1,1,n}, x_{2,1,1}\}$;

$\Pi_3 = \Pi_{1,m,2}^{1,m,n-1} \cup \Pi_{1,2,n}^{1,m-1,n} \cup \Pi_{2,2,1}^{2,m-1,1} \cup \Pi_{2,1,2}^{2,1,n-1}$;

$\Pi_4 = \Pi_{2,2,2}^{2,m-1,n-1} \cup \{x_{1,m,n}, x_{2,m,1}, x_{2,1,n}\}$;

$\Pi_5 = \Pi_{2,m,2}^{2,m,n-1} \cup \Pi_{2,2,n}^{2,m-1,n}$;

$\Pi_6 = \{x_{2,m,n}\}$.

and $\delta_\varphi(x) = 3 - i$ at $x \in \Pi_i$.

At first we will prove that $x_{1,1,1} \overset{\varphi}{\leftarrow} x_{2,1,1}$, i.e., $\varphi(x_{2,1,1})=2$.

Let's assume the reverse: $x_{1,1,1} \overset{\varphi}{\leftarrow} S \overset{\varphi}{\leftarrow} x_{2,1,1}$, and $S \neq \emptyset$.

Consider a case $x_{1,m,n} \notin S$. We have $\delta_\varphi(S) \leq \delta_\varphi(x_{2,1,1}) = 1$ by the Lemma. It is easy to see that for every set X':

$$\delta_\varphi(X') = \frac{1}{|X'|} \sum_{x \in X'} \delta_\varphi(x).$$

As φ is concise with respect to $x_{1,1,1}$, then from $x_{1,m,i} \in S$ follows $x_{1,1,i} \in S$, where $i \in \overline{2, n-1}$ (and from $x_{1,j,n} \in S$ follows $x_{1,j,1} \in S$, where $j \in \overline{2, m-1}$). Therefore, $\delta_\varphi(S) \geq 1$ and $\delta_\varphi(S) = 1$ if and only if

$$S = \Pi_{1,1,1}^{1,m,n} \backslash \{x_{1,1,1}, x_{1,m,n}\}.$$

Let $S \overset{\varphi}{\leftarrow} R \overset{\varphi}{\leftarrow} x_{1,m,n}$ (obviously $x_{2,1,1} \in R$). Easy to see that $\omega(R, x_{1,m,n}) = 0$, $\delta_\varphi(x_{1,m,n}) = -1$, and from the conciseness of φ we have $\delta_\varphi(R) > -1$, which contradicts the Lemma.

Let's now consider the case $x_{1,m,n} \in S$. Then $\Pi_{1,1,1}^{1,m,n} \overset{\varphi}{\leftarrow} \Pi_{2,1,1}^{2,m,n}$ and the subgraphs G_1, G_2 induced with them are ordered minimally at φ. Really, it is easy to see that for every ordering ψ, for which $\Pi_{1,1,1}^{1,m,n} \overset{\psi}{\leftarrow} \Pi_{2,1,1}^{2,m,n}$, we will have $E_\psi(\Pi^{2,m,n}) = m^2n^2 + E_{\psi_1}(G_1) + E_{\psi_2}(G_2)$, where $\psi_1(x) = \psi(x)$ when $x \in \Pi_{1,1,1}^{1,m,n}$ and $\psi_2(x) = \psi(x) - mn$ when $x \in \Pi_{2,1,1}^{2,m,n}$. Therefore, $\varphi \in \phi_{\mathrm{P}2,\mathrm{m,n}}^E$ if and only if $\psi_1 \in \phi_{G_1}^E$, $\psi_2 \in \phi_{G_2}^E$. So G_1, G_2 at φ are ordered minimally. Then from [1] we will have the following. If m ≤ n, then

a) at m > 4: $\Pi_{1,1,1}^{1,\lambda_0,\lambda_0} \overset{\varphi}{\leftarrow} \Pi_{1,1,1}^{1,m,n} \backslash \Pi_{1,1,1}^{1,\lambda_0,\lambda_0} \overset{\varphi}{\leftarrow} \Pi_{2,1,1}^{2,\lambda_0,\lambda_0}$, where $2 \leq \lambda_0 < \frac{1}{2}$m;

b) at m < 4: $\Pi_{1,1,1}^{1,m,1} \overset{\varphi}{\leftarrow} \Pi_{1,1,1}^{1,m,n} \backslash \Pi_{1,1,1}^{1,m,1} \overset{\varphi}{\leftarrow} \Pi_{2,1,1}^{2,m,1}$;

c) at m = 4: the case a) or b) is happened.

It is not difficult to compute:

$$\delta_\varphi\left(\Pi_{1,1,1}^{1,m,n} \backslash \Pi_{1,1,1}^{1,\lambda_0,\lambda_0}\right) = \frac{mn - \lambda_0^2 - 2\lambda_0}{mn - \lambda_0^2} = 1 - \frac{2\lambda_0}{mn - \lambda_0^2} > 0;$$

$\delta_{\varphi}\left(\Pi_{2,1,1}^{2,\lambda_0,\lambda_0}\right) = \dfrac{2\lambda_0 - \lambda_0^2}{\lambda_0^2} = \dfrac{2}{\lambda_0} - 1 \le 0;$

$\delta_{\varphi}\left(\Pi_{1,1,1}^{1,m,n} \backslash \Pi_{1,1,1}^{1,m,1}\right) = \dfrac{m(n-1) - m}{m(n-1)} = \dfrac{n-2}{n-1} > 0;$

$\delta_{\varphi}\left(\Pi_{2,1,1}^{2,m,1}\right) = 0.$

The last relations obviously contradict the Lemma. Therefore, $x_{1,1,1} \overset{\varphi}{\leftarrow} x_{2,1,1}$.

Now let's show, that $x_{1,i,j} \overset{\varphi}{\leftarrow} x_{2,i,j}$ for each i,j ($i \in \overline{1,n}; j \in \overline{1,m}$). We will say that the vertices $x_{1,i,j}, x_{2,i,j}$ are neighbors.

Let's assume the reverse. Let z be a vertex with the smallest number, which does not directly goes behind its neighbor (denote the latter by y).

So we have $y \overset{\varphi}{\leftarrow} S \overset{\varphi}{\leftarrow} z$; $S \neq \emptyset$; $\delta_{\varphi}(y) = \delta_{\varphi}(z) + 2$.

By the definition of z every vertex from $\Pi_{2,1,1}^{2,m,n} \cap S$ directly goes behind its neighbor. Let $\left|\Pi_{2,1,1}^{2,m,n} \cap S\right| = k$ and $y \overset{\varphi}{\leftarrow} M_1 \overset{\varphi}{\leftarrow} N_1 \overset{\varphi}{\leftarrow} \dots \overset{\varphi}{\leftarrow} M_k \overset{\varphi}{\leftarrow} N_k \overset{\varphi}{\leftarrow} M_{k+1} \overset{\varphi}{\leftarrow} z$, where N_i – one pair of neighbors, and $M_j \subset \Pi_{1,1,1}^{1,m,n}$. Then as φ is concise, we have

$$\omega(S, z) = 0; \;\; \omega(y, N_i) = 0; \;\; \omega(M_i, N_i) = 0, \tag{1}$$

at $1 \le i \le j \le k$.

Notice, that y and S cannot be independent of one another. Otherwise, by the Lemma we would have $\delta_{\varphi}(y) \le \delta_{\varphi}(S) \le \delta_{\varphi}(z)$ which would contradict the relation $\delta_{\varphi}(y) = \delta_{\varphi}(z) + 2$. Therefore: $\bigcup M_i \neq \emptyset$.

Let's show, that $\delta_{\varphi}(S) > -1$. Let's assume the reverse: $\delta_{\varphi}(S) \le -1$. Then, as $\delta_{\varphi}(N_i) \ge -1$ for every $i \in \overline{1,k}$, then $\bigcup M_i$ consists of a unique vertex $x_{1,m,n}$. Since $\omega(y, S) \neq 0$, then by (1) we will have $y \in \Pi_3$. Therefore, $\delta_{\varphi}(z) = -2$. But from the conciseness of φ we can conclude, that $x_{1,m,n} \overset{\varphi}{\leftarrow} z$, which contradicts the Lemma. From $\delta_{\varphi}(S) > -1$ we have $\delta_{\varphi}(z) = 0$ ($\delta_{\varphi}(y) = 2$). Notice, that $\delta_{\varphi}(N_i)$ takes values from {-1;0;1}. Let's assume, that $\delta_{\varphi}(N_i) \ge 0$ for each $i \in \overline{1,k}$. Then it is easy to see, that $\delta_{\varphi}(S) > 0$, which is not possible by the Lemma. Therefore, there would be N_i, for which $\delta_{\varphi}(N_i) = -1$.

Let N_p be a pair with the smallest index from $\{N_i\}_{i=\overline{1,k}}$, for which $\delta_{\varphi}(N_i) = -1$. We have $p \ge 1$. We will prove by induction that $M_i = \emptyset$ for all $i \in \overline{2,p}$.

Really, $M_p = \emptyset$ by the Lemma and (1). Let the sets $M_{i+1}, M_{i+2}, \dots, M_p$ be empty. We have

$$M_i \overset{\varphi}{\leftarrow} \bigcup_{j=i+1}^{p} N_j; \;\; \delta_{\varphi}(M_i) \ge 0; \;\; \delta_{\varphi}\left(\bigcup_{j=i+1}^{p} N_j\right) < 0,$$

and by the Lemma we will have $M_i = \emptyset$. Then $y \overset{\varphi}{\leftarrow} M_1 \overset{\varphi}{\leftarrow} \bigcup_{j=1}^{p} N_j$. But $\delta_{\varphi}(y \cup M_1) > 0$, $\delta_{\varphi}\left(\bigcup_{j=1}^{p} N_j\right) < 0$, which contradicts the Lemma. Thus, we obtained that $x_{1,i,j} \overset{\varphi}{\leftarrow} x_{2,i,j}$ for each i,j ($i \in \overline{1,n}; j \in \overline{1,m}$), i.e., vertices of G_1 got odd numbers, while the vertices of G_2 – even numbers.

Let's define layouts φ_1 and φ_2 for the graphs G_1, G_2:

$$\varphi_1\left(\varphi^{-1}(2k-1)\right) = k; \quad \varphi_2\left(\varphi^{-1}(2k)\right) = k;$$

For each $\in \overline{1,mn}$. Then it is easy to see, that

$$E_{\varphi}(P^{2,m,n}) = mn + 2E_{\varphi_1}(G_1) + 2E_{\varphi_2}(G_2). \tag{2}$$

Therefore, φ is minimal if and only if $\varphi \in \phi_{G_i}^{E}$ ($i \in \overline{1,2}$).

This proves the theorem. Substituting the formula of $E(G_i)$ from [1] into (2), at m ≤ n we will have:

$$E_{\varphi}(P^{2,m,n}) = mn + 4\left[-\frac{2}{3}\lambda_0^3 + 2m\lambda_0^2 - \left(m^2 + m - \frac{2}{3}\right)\lambda_0 + n(m^2 + m - 1) - m\right],$$

where $\left\lceil m - \sqrt{\frac{m^2}{2} - \frac{m}{2} + \frac{1}{4}} \right\rceil \leq \lambda_0 \leq \left\lfloor m + \frac{1}{2} - \sqrt{\frac{m^2}{2} - \frac{m}{2} + \frac{1}{4}} \right\rfloor.$

References

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Գծային համարակալումներ եռաչափ ցանցերի համար

Դ. Մուրադյան

Ամփոփում

Գրաֆի մինիմալ համարակալում գտնելու խնդիրը սահմանվում է հետևյալ կերպ: Պահանջվում է գտնել տրված գրաֆի գագաթների այնպիսի տեղաբաշխում թվային առանցքի վրա, որ էջերի երկարությունների գումարը լինի նվազագույն, որտեղ էջի երկարությունը նրան կից գագաթների համարների տարբերության բացարձակ արժեքն է: Այս աշխատանքում նկարագրվում է մինիմալ համարակալումների դասը եռաչափ ցանցերի մի մասնավոր դեպքի՝ երեք պարզ շղթաների դեկարտյան արտադրյալի համար, որոնցից մեկն ունի երկու գագաթ:

Линейные нумерации трехмерных решеток

Д. Мурадян

Аннотация

В работе описывается класс минимальных по длине нумераций частного случая трехмерных решеток – декартого произведения трех простых цепей, когда один из них состоит из двух вершин. Минимальная длина нумерациия графа определяется следующим образом: для данного графа G требуется найти такую линейную нумерацию его вершин, чтобы сумма длин ребер (абсолютное число разности номеров инцидентных ей вершин) была минимальна относительно всевозможных нумераций графа.