Mathematical Problems of Computer Science 46, 44–49, 2016.

A Common Generalization of Dirac's Two Theorems

Carlen M. Mosesyan[1] and Zhora G. Nikoghosyan[2,*]

[1] Kh. Abovyan Armenian State University
[2] Institute for Informatics and Automation Problems of NAS RA
e-mail: mosesyan@list.ru, zhora@ipia.sci.am

Abstract

A theorem is proved including Dirac's two well-known theorems (1952) as particular cases.

Keywords: Hamilton cycle, Longest cycle, Longest path, Minimum degree.

1. Introduction

We consider only undirected graphs with no loops or multiple edges. For a graph G, we use n and c to denote the order and the circumference (the order of a longest cycle) of G. A graph G is hamiltonian if G contains a Hamilton cycle, that is a simple cycle C with $|C| = c = n$. A good reference for any undefined terms is [1].

The earliest two nontrivial lower bounds for the circumference were developed in 1952 due to Dirac [2] in terms of minimum degree δ and p - the order of a longest path in G, respectively.

Theorem A: [2]. *If G is a 2-connected graph, then $c \geq \min\{n, 2\delta\}$.*

Theorem B: [2]. *If G is a 2-connected graph, then $c \geq \sqrt{2p}$.*

In this paper we present a common generalization of Theorem A and Theorem B, including both δ and p in a common relation as parameters.

Theorem 1: *If G is a 2-connected graph, then*

$$c \geq \begin{cases} p, & \text{when} \quad p \leq 2\delta, \\ p-1, & \text{when} \quad 2\delta+1 \leq p \leq 3\delta-2, \\ \sqrt{2p-10+(\delta-\frac{7}{2})^2}+\delta+\frac{1}{2}, & \text{when} \quad p \geq 3\delta-1. \end{cases}$$

Since G is 2-connected, we have $n \geq 3$. If $p \leq 2\delta$, then by Theorem 1, $c \geq p$, implying that $c = p = n$ (G is hamiltonian) and $c = p > \sqrt{2p}$. Next, if $2\delta + 1 \leq p \leq 3\delta - 2$, then by Theorem 1, $c \geq p - 1$. Since $p \geq 2\delta + 1 \geq 5$, we have $c \geq p - 1 \geq 2\delta$ and $c \geq p - 1 > \sqrt{2p}$. Finally, if $p \geq 3\delta - 1$, then

$$\sqrt{2p-10+\left(\delta-\frac{7}{2}\right)^2} \geq \sqrt{2(3\delta-1)-10+\left(\delta-\frac{7}{2}\right)^2} = \delta - \frac{1}{2},$$

*G.G. Nicoghossian (up to 1997)

implying that $c \geq 2\delta$ (by Theorem 1) and

$$\left(\delta+\frac{1}{2}\right)\sqrt{2p-10+\left(\delta-\frac{7}{2}\right)^2}+\delta^2-3\delta+\frac{5}{4}$$

$$\geq \left(\delta+\frac{1}{2}\right)\left(\delta-\frac{1}{2}\right)+\delta^2-3\delta+\frac{5}{4}=(\delta-1)(2\delta-1)>0.$$

Observing that the inequality

$$\left(\delta+\frac{1}{2}\right)\sqrt{2p-10+\left(\delta-\frac{7}{2}\right)^2}+\delta^2-3\delta+\frac{5}{4}>0$$

is equivalent to

$$\sqrt{2p-10+\left(\delta-\frac{7}{2}\right)^2}+\delta+\frac{1}{2}>\sqrt{2p},$$

we conclude (by Theorem 1) that $c > \sqrt{2p}$.

Thus, Theorem 1 yields Theorem A and is stronger than Theorem B.

To show that Theorem 1 is best possible in a sense, observe first that in general, $p \geq c$, that is $c = p$ when $p \leq 2\delta$, implying that the bound $c \geq p$ in Theorem 1 cannot be replaced by $c \geq p+1$. On the other hand, the graph $K_{\delta,\delta+1}$, where $p = 2\delta+1$ and $c = 2\delta = p-1$ shows that the condition $p \leq 2\delta$ cannot be relaxed to $p \leq 2\delta+1$. In addition, the graph $K_{\delta,\delta+1}$, where $c = p$, shows that the bound $c \geq p-1$ (when $2\delta+1 \leq p \leq 3\delta-2$) cannot be replaced by $c \geq p$. Further, the graph $K_2+3K_{\delta-1}$, where $n = p = 3\delta-1$ and $c = 2\delta \leq p-2$ shows that the condition $p \leq 3\delta-2$ cannot be relaxed to $p \leq 3\delta-1$. Finally, the same graph $K_2+3K_{\delta-1}$, where $p = 3\delta-1$ and

$$c=2\delta=\sqrt{2p-10+\left(\delta-\frac{7}{2}\right)^2}+\delta+\frac{1}{2},$$

shows that the bound $\sqrt{2p-10+(\delta-\frac{7}{2})^2}+\delta+\frac{1}{2}$ in Theorem 1 cannot be improved to $\sqrt{2p-10+(\delta-\frac{7}{2})^2}+\delta+1$.

For a special case when $2\delta+1 \leq p \leq 3\delta-2$, we use the result of Ozeki and Yamashita [3].

Theorem C: [3]. *Let G be a 2-connected graph. Then either*

(i) $c \geq p-1$ or

(ii) $c \geq 3\delta-3$ or

(iii) $\kappa = 2$ and $p \geq 3\delta-1$.

2. Notation and Preliminaries

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges - by $E(G)$. The neighborhood of a vertex $x \in V(G)$ will be denoted by $N(x)$. We use $d(x)$ to denote $|N(x)|$.

Paths and cycles in a graph G are considered as subgraphs of G. If Q is a path or a cycle, then the order of Q, denoted by $|Q|$, is $|V(Q)|$. We write a cycle Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. For $x \in V(Q)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{Q}$ by x^{+h} and x^{-h}, respectively. We abbreviate x^{+1} and x^{-1} by x^+ and x^-, respectively. For

$U \subseteq V(Q)$, $U^+ = \{u^+ | u \in U\}$ and $U^- = \{u^- | u \in U\}$. We say that vertex z_1 precedes vertex z_2 on $\overrightarrow{Q}$ if z_1, z_2 occur on $\overrightarrow{Q}$ in this order, and indicate this relationship by $z_1 \prec z_2$. We will write $z_1 \preceq z_2$ when either $z_1 = z_2$ or $z_1 \prec z_2$.

Let $P = x\overrightarrow{P}y$ be a path. A vine on P is a set

$$\{L_i = x_i \overrightarrow{L}_i y_i : 1 \leq i \leq m\}$$

of internally-disjoint paths such that

(a) $V(L_i) \cap V(P) = \{x_i, y_i\}$ $(i = 1, ..., m)$,
(b) $x = x_1 \prec x_2 \prec y_1 \preceq x_3 \prec y_2 \preceq x_4 \prec ... \preceq x_m \prec y_{m-1} \prec y_m = y$ on P.

The Vine Lemma: [4]. *Let G be a k-connected graph and P a path in G. Then there are $k-1$ pairwise-disjoint vines on P.*

The next three lemmas are crucial for the proof of Theorem 1.

Lemma 1: *Let G be a connected graph and $P = x\overrightarrow{P}y$ a longest path in G.*

(i) If $xz, yz^- \in E(G)$ for some $z \in V(x^+\overrightarrow{P}y)$, then $c = p = n$, that is G is hamiltonian.

(ii) If $d(x) + d(y) \geq p$, then $c = p = n$.

(iii) Let $yz_1, xz_2 \in E(G)$ for some $z_1, z_2 \in V(P)$ with $x \prec z_1 \prec z_2 \prec y$ and $|z_1\overrightarrow{P}z_2| \geq 3$. If $xz, yz \notin E(G)$ for each $z \in V(z_1^+\overrightarrow{P}z_2^-)$ and $d(x) + d(y) \geq p + 3 - |z_1\overrightarrow{P}z_2|$, then $c = p$.

Lemma 2: *Let G be a 2-connected graph and $\{L_1, L_2, ..., L_m\}$ be a vine on a longest path of G. Then*

$$c \geq \frac{2p - 10}{m + 1} + 4.$$

Lemma 3: *Let G be a connected graph and $\{L_1, L_2, ..., L_m\}$ be a vine on a longest path $P = x\overrightarrow{P}y$ of G. Then either $c = p$ or $c \geq d(x) + d(y) + m - 2$.*

3. Proofs

Proof of Lemma 1: (i) Let $xz, yz^- \in E(G)$ for some $z \in V(x^+\overrightarrow{P}y)$. Then $c \geq |xz\overrightarrow{P}yz^-\overleftarrow{P}x| = p$. If $V(G) = V(P)$, then clearly $c = p$. Otherwise, recalling that G is connected, we can form a path longer that P, a contradiction.

(ii) Let $d(x) + d(y) \geq p$. If $xz, yz^- \in E(G)$ for some $z \in V(x^+\overrightarrow{P}y)$, then we can argue as in (i). Otherwise $N(x) \cap N^+(y) = \emptyset$. Observing also that $x \notin N(x) \cup N^+(y)$, we get

$$p \geq |N(x)| + |N^+(y)| + |\{x_1\}|$$

$$= |N(x)| + |N(y)| + 1 = d(x) + d(y) + 1,$$

contradicting the hypothesis.

(iii) Assume the contrary, that is $c \leq p - 1$. Then by (i), $N(x) \cap N^+(y) = \emptyset$. Clearly, $x \notin N(x) \cup N^+(y)$. Further, by the hypothesis,

$$V(z_1^{+2}\overrightarrow{P}z_2^-) \cap (N(x) \cup N^+(y)) = \emptyset,$$

implying that

$$p \geq |\{x\}| + |N(x)| + |N^+(y)| + |V(z_1^{+2}\overrightarrow{P}z_2^-)|$$

$$= d(x) + d(y) + |z_1 \overrightarrow{P} z_2| - 2,$$

contradicting the hypothesis. Thus, $c = p$. Lemma 1 is proved. ■

Proof of Lemma 2: Let $P = x\overrightarrow{P}y$ be a longest path in G. Put

$$L_i = x_i \overrightarrow{L}_i y_i \quad (i = 1, ..., m), \quad A_1 = x_1 \overrightarrow{P} x_2, \quad A_m = y_{m-1} \overrightarrow{P} y_m,$$

$$A_i = y_{i-1} \overrightarrow{P} x_{i+1} \quad (i = 2, 3, ..., m-1), \quad B_i = x_{i+1} \overrightarrow{P} y_i \quad (i = 1, ..., m-1),$$

$$|A_i| - 1 = a_i \quad (i = 1, ..., m), \quad |B_i| - 1 = b_i \quad (i = 1, ..., m-1).$$

By combining appropriate L_i, A_i, B_i, we can form the following cycles:

$$Q_1 = \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{m} L_i,$$

$$Q_2 = \bigcup_{i=1}^{m-1} A_i \cup B_{m-1} \cup \bigcup_{i=1}^{m-1} L_i,$$

$$Q_3 = \bigcup_{i=2}^{m} A_i \cup B_1 \cup \bigcup_{i=2}^{m} L_i,$$

$$R_i = B_i \cup A_{i+1} \cup B_{i+1} \cup L_{i+1} \quad (i = 1, ..., m-2).$$

Since $|L_i| \geq 2$ $(i = 1, ..., m)$, we have

$$c \geq |Q_1| = \sum_{i=1}^{m} a_i + \sum_{i=1}^{m} (|L_i| - 1) \geq \sum_{i=1}^{m} a_i + m,$$

$$c \geq |Q_2| = b_{m-1} + \sum_{i=1}^{m-1} a_i + \sum_{i=1}^{m-1} (|L_i| - 1) \geq b_{m-1} + \sum_{i=1}^{m-1} a_i + m - 1,$$

$$c \geq |Q_3| = b_1 + \sum_{i=2}^{m} a_i + \sum_{i=2}^{m} (|L_i| - 1) \geq b_1 + \sum_{i=2}^{m} a_i + m - 1,$$

$$c \geq |R_i| = b_i + a_{i+1} + b_{i+1} + |L_{i+1}| - 1$$
$$\geq b_i + a_{i+1} + b_{i+1} + 1 \quad (i = 1, ..., m-2).$$

By summing, we get

$$(m+1)c \geq \left(2\sum_{i=1}^{m} a_i + 2\sum_{i=1}^{m-1} b_i \right) + 2\sum_{i=2}^{m-1} a_i + 4m - 4$$

$$\geq 2\left(\sum_{i=1}^{m} a_i + \sum_{i=1}^{m-1} b_i + 1 \right) + 4m - 6 = 2p + 4m - 6,$$

implying that

$$c \geq \frac{2p - 10}{m+1} + 4.$$

Lemma 2 is proved. ■

Proof of Lemma 3: If $m = 1$, then $xy \in E(G)$ and by Lemma 1(i), $c = p$. Let $m \geq 2$. Put $L_i = x_i \overrightarrow{L}_i y_i$ $(i = 1, ..., m)$ and let

$$A_i, \ B_i, \ a_i, \ b_i, \ Q_i$$

be as defined in the proof of Lemma 2.

Case 1: $m = 2$.

Assume without loss of generality that L_1 and L_2 are chosen so as to minimize b_1. This means that $N(x) \cup N(y) \subseteq V(A_1 \cup A_2)$. By Lemma 1$(iii)$, either $c = p$ or $d(x) + d(y) \leq p + 2 - |z_1 \overrightarrow{P} z_2| = p + 1 - b_1$. If $c = p$, then we are done. Let $d(x) + d(y) \leq p + 1 - b_1$, that is $p \geq d(x) + d(y) + b_1 - 1$. Then $p = a_1 + a_2 + b_1 + 1 \geq d(x) + d(y) + b_1 - 1$, implying that

$$c \geq |Q_1| = a_1 + a_2 + 2 \geq d(x) + d(y) = d(x) + d(y) + m - 2.$$

Case 2: $m = 3$.

Let $xz_1, yz_2 \in E(G)$ for some $z_1, z_2 \in V(P)$. If $z_2 \prec z_1$ then $\{xz_1, yz_2\}$ is a vine consisting of two paths (edges) and we can argue as in Case 1. Otherwise we have

$$N(x) \subseteq V(A_1 \cup A_2), \quad N(y) \subseteq V(A_2 \cup A_3)$$

and $z_1 \preceq z_2$ for each $z_1 \in N(x)$ and $z_2 \in N(y)$. Therefore, $a_1 + a_2 + a_3 \geq d(x) + d(y) - 2$ and

$$\begin{aligned} c &\geq |Q_1| = a_1 + a_2 + a_3 + 3 \\ &\geq d(x) + d(y) + 1 = d(x) + d(y) + m - 2. \end{aligned}$$

Case 3: $m \geq 4$.

Choose $\{L_1, ..., L_m\}$ so as to minimize m. Then clearly

$$N(x) \subseteq V(A_1 \cup A_2), \quad N(y) \subseteq V(A_{m-1} \cup A_m)$$

and $z_1 \prec z_2$ for each $z_1 \in N(x)$ and $z_2 \in N(y)$. Observing also that

$$a_1 + a_2 \geq d(x) - 1, \quad a_{m-1} + a_m \geq d(y) - 1,$$

we get

$$\begin{aligned} c \geq |Q_1| &= \sum_{i=1}^{m} a_i + m = (a_1 + a_2 + a_{m-1} + a_m) + \sum_{i=3}^{m-2} a_i + m \\ &\geq d(x) + d(y) - 2 + \sum_{i=3}^{m-2} a_i + m \geq d(x) + d(y) + m - 2. \end{aligned}$$

Lemma 3 is proved. ∎

Proof of Theorem 1: Let $P = x\overrightarrow{P}y$ be a longest path in G.

Case 1: $p \leq 2\delta$.

If $xy \in E(G)$, then by Lemma 1(i), $c = p$. Let $xy \notin E(G)$. Then $d(x) + d(y) \geq 2\delta \geq p$ and by Lemma 1(ii), $c = p$.

Case 2: $2\delta + 1 \leq p \leq 3\delta - 2$.

If $c \geq 3\delta - 3$, then $c \geq p + 2 - 3 = p - 1$. Next, if $\kappa = 2$ and $p \geq 3\delta - 1$, then $p \geq 3\delta - 1 \geq p + 1$, a contradiction. By Theorem C, $c \geq p - 1$.

Case 3: $p \geq 3\delta - 1$.

Since G is 2-connected, there is a vine $\{L_1, ..., L_m\}$ on P. By Lemma 3, $m \leq c - d(x) - d(y) + 2 \leq c - 2\delta + 2$. Using Lemma 2, we get

$$c \geq \frac{2p - 10}{m + 1} + 4 \geq \frac{2p - 10}{c - 2\delta + 3} + 4,$$

implying that

$$c \geq \sqrt{2p - 10 + \left(\delta - \frac{7}{2}\right)^2} + \delta + \frac{1}{2}.$$

Theorem 1 is proved. ∎

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[2] G. A. Dirac, "Some theorems on abstract graphs", *Proc. London, Math. Soc.*, vol. 2, pp. 69-81, 1952.

[3] K. Ozeki and T. Yamashita, "Length of longest cycles in a graph whose relative length is at least two", *Graphs and Combinatorics*, vol. 28, pp. 859-868, 2012.

[4] J. A. Bondy, *Basic Graph Theory: Paths and circuits*, Handbook of Combinatorics, Elsevier, Amsterdam, vol. 1,2, 1990.

Submitted 01.07.2016, accepted 02.11.2016.

Դիրակի երկու թեորեմների ընդհանրացում

Կ. Մոսեսյան և Ժ. Նիկողոսյան

Ամփոփում

Ապացուցվում է մի թեորեմ, որն ընդգրկում է 1952-ին Դիրակի կողմից ստացված երկու հայտնի թեորեմները որպես մասնավոր դեպքեր:

Обобщение двух теорем Дирака

К. Мосесян и Ж. Никогосян

Аннотация

Доказывается одна теорема, которая включает две известные теоремы Дирака, полученные в 1952 г., как частные случаи.