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Spanning Trees with few Branch and End Vertices

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Abstract

For a graph G, let σ_2 be the minimum degree sum of two nonadjacent vertices in G. A vertex of degree one in a tree is called an end vertex and a vertex of degree at least three is called a branch vertex. We consider: $(*)$ connected graphs on n vertices such that $\sigma_2 \geq n-k+1$ for some positive integer k. In 1976, it was proved (by the author) that every graph satisfying $(*)$, has a spanning tree with at most k end vertices. In this paper we first show that every graph satisfying $(*)$, has a spanning tree with at most $k+1$ branch and end vertices altogether. The next result states that every graph satisfying $(*)$, has a spanning tree with at most $\frac{1}{2}(k-1)$ branch vertices. The third result states that every graph satisfying $(*)$, has a spanning tree with at most $\frac{3}{2}(k-1)$ degree sum of branch vertices. All results are sharp.

Keywords: Spanning tree, End vertex, k-ended tree, Branch vertex, Degree sum of the branch vertices, Ore-type condition.

1. Introduction

Throughout this article we consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by - $E(G)$. A good reference for any undefined terms is [1].

For a graph G, we use n and α to denote the order (the number of vertices) and the independence number of G, respectively. If $\alpha \geq k$ for some integer k, let σ_k be the minimum degree sum of an independent set of k vertices; otherwise we let $\sigma_k = +\infty$. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. We use $d_G(v)$ to denote the number of neighbors of a vertex v in G, called the degree of v in G. The minimum degree in G is denoted by δ.

If Q is a path or a cycle in a graph G, then the order of Q, denoted by $|Q|$, is $|V(Q)|$. The graph G is hamiltonian if G contains a Hamilton cycle, i.e., a cycle containing every vertex of G.

We write a path Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. We use x^+ to denote the successor, and x^- the predecessor, of a vertex $x \in V(Q)$. For $X \subseteq V(Q)$, we define $X^+ = \{x^+ : x \in X\}$ and $X^- = \{x^- : x \in X\}$.

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A vertex of degree one is called an end-vertex, and an end-vertex of a tree is usually called a leaf. The set of end-vertices of G is denoted by $End(G)$. A branch vertex of a tree is a vertex of degree at least three. The set of branch vertices of a tree T will be denoted by $B(T)$. For a positive integer k, a tree T is said to be a k-ended tree if $|End(T)| \leq k$. A Hamilton path is a spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree.

In 1952, Dirac [2] proved that every graph with $\delta \geq n/2$ has a Hamilton cycle. The degree sum version of this result was proved in 1960 due to Ore [3]: every graph with $\sigma_2 \geq n$ has a Hamilton cycle.

The analogs of these two classical results for Hamilton paths follow easily.

Theorem A: [2]. *Every graph with $\delta \geq (n-1)/2$ has a Hamilton path.*

Theorem B: [3]. *Every graph with $\sigma_2 \geq n-1$ has a Hamilton path.*

The next result took a different approach due to Chvátal and Erdős [4] based on connectivity and independence number: every k-connected ($k \geq 1$) graph with $\alpha \leq k$ has a Hamilton cycle. The Hamilton path version of this result can be formulated as follows.

Theorem C: [4]. *Every k-connected ($k \geq 1$) graph with $\alpha \leq k+1$ has a Hamilton path.*

A Hamilton path can be regarded as a spanning tree with exactly two leaves, a spanning tree with no branch vertex, or a spanning tree with maximum degree two. Therefore, as one of generalized problems of a Hamilton path problem, it is natural to look for conditions which ensure the existence of a spanning tree with few leaves, few branch vertices or bounded maximum degree motivated from optimization aspects with various applications.

In this paper we consider tree problems arising in the context of optical and centralized terminal networks: (i) finding a spanning tree of G with the minimum number of end vertices, (ii) finding a spanning tree with the minimum number of branch vertices and (iii) finding a spanning tree of G such that the degree sum of the branch vertices is minimized, motivated by network design problems where junctions are significantly more expensive than simple end- or through-nodes, and are, thus, to be avoided.

All these problems are NP-hard because they contain the Hamilton path problem as a particular case.

The constraint on the number of end vertices arises because the software and hardware associated to each terminal differs accordingly with its position in the tree. Usually, the software and hardware associated to a "degree-l" terminal is cheaper than the software and hardware used in the remaining terminals because for any intermediate terminal j one needs to check if the arrival message is destined to that node or to any other node located after node j. As a consequence, that particular terminal needs software and hardware for message routing. On the other hand, such equipment is not needed in "degree-l" terminals. Assuming that the hardware and software for message routing in the nodes is already available, the above discussion motivates a constraint stating that a tree solution has to contain exactly a certain number of "degree-l" terminals.

A different situation, resulting from a new technology allowing a switch to replicate the signal by splitting light. A light-tree connects one node to a set of other nodes in the network - allowing multicast communication from the source to a set of destinations (including the possibility of the set of destinations consisting of all other nodes). The switches which correspond to the nodes of degree greater than two have to be able to split light (except for the source of the multicast, which can transmit to any number of neighbors). Typical optical networks will have a limited number of these more sophisticated switches, and one has to position them in such a way that all possible multicasts can be performed. Thus, we are lead to the problem of finding spanning trees with as few branch vertices as possible.

In 1971, Las Vergnas [5] gave a degree condition that guarantees that any forest in G of limited size and with a limited number of leaves can be extended to a spanning tree of G with a limited number of leaves in an appropriate sense. This result implies as a corollary a degree sum condition for the existence of a tree with at most k leaves including Theorem B as a special case for $k = 1$.

Theorem D: [6], [5], [7]. *Let G be a connected graph with $\sigma_2 \geq n - k + 1$ for some positive integer k. Then G has a spanning k-ended tree.*

However, Theorem D was first openly formulated and proved in 1976 by the author [6]. Later, it was reproved in 1998 by Broersma and Tuinstra [7].

The next generalization contains Theorem C as a special case ($k = 1$) due to Win [8].

Theorem E: [8]. *Let G be an s-connected graph with $\alpha \leq s + k - 1$ for some integer $k \geq 1$. Then G has a spanning k-ended tree.*

One of the interest in the existence of spanning trees with bounded number of branch vertices arises in the realm of multicasting in optical networks.

Gargano, Hammar, Hell, Stacho and Vaccaro [9] proved the following.

Theorem F: [9]. *Every connected graph with $\sigma_3 \geq n - 1$ has a spanning tree with at most one branch vertex.*

Flandrin et al. [10] posed the following conjecture.

Conjecture A: [10]. *If G is a connected graph with $\sigma_{k+3} \geq n - k$ for some positive integer k, then G has a spanning tree with at most k branch vertices.*

Recently, Matsuda, Ozeki and Yamashita [11] established an upper bound for the independence number α which implies the existence of a spanning tree with bounded number of branch vertices in connected claw-free graphs.

Theorem G: [11]. *Let k be a non-negative integer. A connected claw-free graph G has a spanning tree with at most k branch vertices if $\alpha \leq 2k + 2$.*

In this paper we present a sharp Ore-type condition for the existence of spanning trees in connected graphs with bounded total number of branch and end vertices improving Theorem D by incorporating the number of branch vertices as a parameter.

Theorem 1:. *Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree T with at most $k - |B(T)| + 1$ end vertices.*

Let G be the complete bipartite graph $K_{\delta,\delta+k-1}$ of order $n = 2\delta + k - 1$ and minimum degree δ, where $k \geq 3$. Clearly, $\sigma_2(G) = 2\delta = n - k + 1$. By Theorem 1, G has a spanning tree T with $|End(T)| \leq k - b + 1$. Observing that T is not $(k-1)$-ended, that is $|End(T)| \geq k$, we have $b \leq 1$. On the other hand, we have $b \geq 1$, since $|End(T)| \geq k \geq 3$, which implies $b = 1$. This means that T is not $(k - b)$-ended and consequently, Theorem 1 is sharp for each $k \geq 3$.

The next result follows from Theorem 1 providing a sharp Ore-type condition for the existence of spanning trees in connected graphs with few branch vertices.

Theorem 2: *Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree with at most $(k - 1)/2$ branch vertices.*

The third result provides an Ore-type condition for the existence of spanning trees in connected graphs with bounded degree sum of the branch vertices.

Theorem 3: *Let G be a connected graph of order n. If $\sigma_2 \geq n - k + 1$ for some positive integer k, then G has a spanning tree with at most $\frac{3}{2}(k-1)$ degree sum of the branch vertices.*

Let G be a graph (tree) obtained from the path $v_0v_1...v_bv_{b+1}$ by adding new vertices

$u_1, ..., u_b$ and the edges u_iv_i $(i = 1, ..., b)$. Clearly, $n = 2b + 2$ and $\sigma_2 = 2 = n - (2b + 1) + 1$. Since $|B(G)| = b$, the bound $(k - 1)/2$ in Theorem 2 is sharp. Further, since $\sum_{i=1}^{b} d(v_i) = \frac{3}{2}(k - 1)$, the bound $\frac{3}{2}(k - 1)$ in Theorem 3 is sharp as well.

Theorems 1,2,3 were announced in 2015 [12] and Theorem 1 was proved independently by Saito and Sano [13].

2. Proof of Theorem 1

Proof of Theorem 1: Let G be a connected graph with $\sigma_2 \geq n - k + 1$ and let T be a spanning tree in G. Assume that

$(a1)$ T is chosen so that $|End(T)|$ is as small as possible.

Put $End(T) = \{\xi_1, ..., \xi_f\}$. Let $\overrightarrow{P_2} = \xi_1\overrightarrow{P_2}\xi_2$ be the unique path in T with end vertices ξ_1 and ξ_2. Further, assume that

$(a2)$ T is chosen so that P_2 is as long as possible, subject to $(a1)$.

Put $|B(T)| = b$. If $f = 2$ then P_2 is a 2-ended spanning tree (Hamilton path) in G with $|B(P_2)| = b = 0$, implying that $f = 2 \leq k + 1 = k - b + 1$.

Now let $f \geq 3$, that is $b \geq 1$.

Claim 1: *If P is a Hamilton path in $G[V(P_2)]$ with end vertices x, y, then $N(x) \cup N(y) \subseteq V(P_2)$.*

Proof: Assume the contrary and assume w.l.o.g. that $N(x) \not\subseteq V(P_2)$. Put $T' = T - E(P_2) + E(P)$. Clearly, T' is an f-ended spanning tree in G and $xv \in E(G)$ for some $v \in V(G - P)$. Let C be the unique cycle in $T' + xv$ and let vv' be the unique edge on C with $v' \neq x$. Then $T' + xv - vv'$ is an f-ended spanning tree in G, contradicting $(a2)$. △

By Claim 1, $N(\xi_1) \cup N(\xi_2) \subseteq V(P_2)$. If $N(\xi_1) \cap N^+(\xi_2) \neq \emptyset$ then clearly, $G[V(P_2)]$ has a Hamilton cycle. Since $b \geq 1$, $G[V(P_2)]$ has a Hamilton path with end vertex x such that $N(x) \not\subseteq V(P_2)$, contradicting Claim 1. Hence, $N(\xi_1) \cap N^+(\xi_2) = \emptyset$. Observing also that $\xi_1 \notin N(\xi_1) \cup N^+(\xi_2)$ and $N^+(\xi_2) \subseteq V(P_2)$, we get

$$|P_2| \geq |N(\xi_1)| + |N^+(\xi_2)| + |\{\xi_1\}|$$

$$= d(\xi_1) + d(\xi_2) + 1 \geq \sigma_2 + 1. \quad (1)$$

For each $i \in \{3, ..., f\}$, let $\overrightarrow{P_i} = \xi_i\overrightarrow{P_i}z_i$ be the unique path in T between ξ_i and the nearest vertex z_i of P_2. Clearly, $z_i \in B(T)$ $(i = 3, ..., f)$.

Case 1: $|P_i| = 2$ $(i = 3, ..., f)$.

It follows that $B(T) \subseteq V(P_2)$. If $b = 1$, then by (1), $|P_2| \geq \sigma_2 + b$ and therefore,

$$f = |\{\xi_3, ..., \xi_f\}| + 2 = n - |P_2| + 2$$

$$\leq n - \sigma_2 - b + 2 \leq k - b + 1.$$

Let $b \geq 2$ and let $x_1, ..., x_b$ be the elements of $B(T)$, occurring on $\overrightarrow{P_2}$ in a consecutive order. Assume w.l.o.g. that $x_1 = z_3$. Further, assume that

$(a3)$ T is chosen so that $d_T(x_1)$ is as small as possible, subject to $(a1)$ and $(a2)$.

If $\xi_3 v_1 \in E(G)$ for some $v_1 \in V(x_1^+ \overrightarrow{P_2} \xi_2)$, then $T + \xi_3 v_1 - \xi_3 x_1$ is an f-ended tree, contradicting (a3). Hence, we can assume that $N(\xi_3) \subseteq V(\xi_1 \overrightarrow{P_2} x_1)$, that is

$$(N(\xi_3) - z_3) \cap B(T) = \emptyset. \tag{2}$$

Next, if $N^-(\xi_1) \cap (N(\xi_3) - z_3)$ has an element v_2, then

$$v_2 \overleftarrow{P_2} \xi_1 v_2^+ \overrightarrow{P_2} \xi_2$$

is a Hamilton path in $G[V(P_2)]$ with end vertex v_2 such that $N(v_2) \not\subseteq V(P_2)$, contradicting Claim 1. Hence,

$$N^-(\xi_1) \cap (N(\xi_3) - z_3) = \emptyset. \tag{3}$$

Finally, if $N^-(\xi_1) \cap B(T) \neq \emptyset$, that is $\xi_1 z_i^+ \in E(G)$ for some $i \in \{3, ..., f\}$, then

$$z_i \overleftarrow{P_2} \xi_1 z_i^+ \overrightarrow{P_2} \xi_2$$

is a Hamilton path in $G[V(P_2)]$ with end vertex z_i such that $N(z_i) \not\subseteq V(P_2)$, again contradicting Claim 1. Hence,

$$N^-(\xi_1) \cap B(T) = \emptyset. \tag{4}$$

Using (2), (3), (4) and observing that $\xi_2 \notin N^-(\xi_1) \cup (N(\xi_3) - z_3) \cup B(T)$, we get

$$|V(P_2)| \geq |N^-(\xi_1)| + |N(\xi_3) - z_3| + |B(T)| + |\{\xi_2\}|$$

$$\geq d(\xi_1) + d(\xi_3) + b \geq \sigma_2 + b,$$

implying that

$$f = |\{\xi_3, ..., \xi_f\}| + 2 = n - |V(P_2)| + 2$$

$$\leq n - \sigma_2 - b + 2 \leq k - b + 1.$$

Case 2: $|P_i| \geq 3$ for some $i \in \{3, ..., f\}$, say $i = 3$.

Case 2.1: $N^-(\xi_1) \cap N^+(\xi_2) \neq \emptyset$.

It follows that $\xi_1 w^+, \xi_2 w^- \in E(G)$ for some $w \in N^-(\xi_1) \cap N^+(\xi_2)$. If $z_3 = w$ then

$$w \overleftarrow{P_2} \xi_1 w^+ \overrightarrow{P_2} \xi_2$$

is a Hamilton path in $G[V(P_2)]$ with end vertex w such that $N(w) \not\subseteq V(P_2)$, contradicting Claim 1. Hence $z_3 \neq w$. Assume w.l.o.g. that $z_3 \in V(\xi_1 \overrightarrow{P_2} w^-)$. Put

$$T' = T + \xi_1 w^+ + \xi_2 w^- - z_3 z_3^- - w w^-.$$

Clearly, T' is a spanning f-ended tree in G and

$$\xi_3 \overrightarrow{P_3} z_3 \overrightarrow{P_2} w^- \xi_2 \overleftarrow{P_2} w^+ \xi_1 \overrightarrow{P_2} z_3^-$$

is a path in T' longer than P_2, contradicting (a2).

Case 2.2: $N^-(\xi_1) \cap N^+(\xi_2) = \emptyset$.

Put

$$B_1 = V(P_2) \cap B(T), \; B_2 = B(T) - B_1.$$

Using Claim 1, it is easy to see that

$$N^-(\xi_1) \cap B_1 = N^+(\xi_2) \cap B_1 = \emptyset.$$

Observing also that $N^-(\xi_1) \cup N^+(\xi_2) \subseteq V(P_2)$, we get

$$|P_2| \geq |N^-(\xi_1)| + |N^+(\xi_2)| + |B_1|$$

$$= d(\xi_1) + d(\xi_2) + |B_1| \geq \sigma_2 + |B_1| \geq n - k + 1 + |B_1|.$$

Then

$$n \geq |P_2| + |B_2| + |\{\xi_3, ..., \xi_f\}|$$

$$\geq n - k + 1 + |B_1| + |B_2| + f - 2 = n - k + b + f - 1,$$

implying that $f \leq k - b + 1$. ■

Proof of Theorem 2: By Theorem 1, G has a spanning tree T with $|End(T)| \leq k - b + 1$, where $b = |B(T)|$. On the other hand, it is not hard to see that $|End(T)| \geq b + 2$, implying that $b \leq (k-1)/2$. ■

Proof of Theorem 3: By Theorem 1 and Theorem 2, G has a spanning tree T with $f = |End(T)| \leq k - b + 1$ and $b \leq (k-1)/2$, where $b = |B(T)|$. Let $d_1, d_2, ..., d_b$ be the degrees of branch vertices of T. Observing that

$$f = \sum_{i=1}^{b}(d_i - 2) + 2,$$

we get

$$\sum_{i=1}^{b} d_i \leq k + b - 1 \leq \frac{3}{2}(k-1). \qquad ■$$

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Կմախքային ծառեր քիչ ճյուղային և կախված գագաթներով

Ժ. Նիկողոսյան

Ամփոփում

G գրաֆի համար σ_2-ով նշանակվում է G-ի ոչ հարևան գագաթների աստիճանների նվազագույն գումարը։ Ծառի մեկ աստիճան ունեցող գագաթը կոչվում է կախված գագաթ, իսկ առնվազն երեք աստիճան ունեցող գագաթը՝ ճյուղային գագաթ։ Աշխատանքում դիտարկվում են միայն $(*)$ n գագաթանի կապակցված գրաֆներ, որոնք բավարարում են $\sigma_2 \geq n-k+1$ պայմանին ինչ-որ մի k բնական թվի համար։ 1976-ին ապացուցվել է (հեղինակի կողմից), որ $(*)$-ին բավարարող կամայական գրաֆ ունի ամենաշատը k կախված գագաթ ունեցող կմախքային ծառ։ Ներկա աշխատանքում նախ ցույց է տրվում, որ $(*)$-ին բավարարող կամայական գրաֆ ունի կմախքային ծառ, որի կախված և ճյուղային գագաթների ընդհանուր քանակը չի գերազանցում $(k+1)$-ը։ Երկրորդ արդյունքը պնդում է, որ $(*)$-ին բավարարող կամայական գրաֆ ունի կմախքային ծառ ամենաշատը $\frac{1}{2}(k-1)$ ճյուղային գագաթներով։ Ըստ երրորդ արդյունքի, $(*)$-ին բավարարող կամայական գրաֆ ունի կմախքային ծառ, որի ճյուղային գագաթների աստիճանների գումարը չի գերազանցում $\frac{3}{2}(k-1)$-ը։ Բոլոր երեք գնահատականները հասանելի են։

Каркасы с меньшим числом висячих и Br-вершин

Ж. Никогосян

Аннотация

Для графа G через σ_2 обозначается минимальная сумма степеней двух несмежных вершин. Вершина в дереве называется висячей, если имеет степень 1; и называется Br-вершиной, если имеет степень по меньшей мере 3. В работе рассматриваются только $(*)$ n-вершинные связные графы, удовлетворяющие условию $\sigma_2 \geq n-k+1$ для некоторого натурального числа k. В 1976 году

была доказана (автором), что произвольный граф удовлетворяющий условию $(*)$, имеет каркас с не более чем k висячими вершинами. В настоящей работе доказывается, что всякий граф удовлетворяющий условию $(*)$, имеет каркас, где общее число висячих и Br-вершин не превосходит $k+1$. Второй результат утверждает, что всякий граф, удовлетворяющий условию $(*)$, имеет каркас, где число Br-вершин не превосходит $\frac{1}{2}(k-1)$. Третий результат утверждает, что всякий граф, удовлетворяющий условию $(*)$, имеет каркас, где сумма степеней Br-вершин не превосходит $\frac{3}{2}(k-1)$. Все три оценки достигаемы.