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On pre-Hamiltonian Cycles in Balanced Bipartite Digraphs

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Abstract

Let D be a strongly connected balanced bipartite directed graph of order $2a \geq 10$. Let x, y be distinct vertices in D. $\{x, y\}$ dominates a vertex z if $x \rightarrow z$ and $y \rightarrow z$; in this case, we call the pair $\{x, y\}$ dominating. In this paper we prove:

If $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$, then either the underlying graph of D is 2-connected or D contains a cycle of length $2a-2$ or D is isomorphic to one digraph of order ten.

Keywords: Digraphs, Cycles, Hamiltonian cycles, Bipartite balanced digraph, Pancyclic, Even pancyclic.

1. Introduction

We consider directed graphs (digraphs) in the sense of [1]. A cycle of a digraph D is called Hamiltonian if it contains all the vertices of D. For convenience of the reader terminology and notations will be given in details in section 2. A digraph D of order n is Hamiltonian if it contains a Hamiltonian cycle and pancyclic if it contains cycles of every length k, $3 \leq k \leq n$. For general digraphs there are several sufficient conditions for existence of Hamiltonian cycles in digraphs. In this paper, we will be concerned with the degree conditions.

The well-known and classical are Ghouila-Houri's, Nash-Williams', Woodall's, Meyniel's and Thomas
sen's theorems (see, e.g., [2]- [6]). There are analogous results of the above-mentioned theorems for the pancyclicity of digraphs (see, e.g., [7-12]). Each of theorems ([2]-[6]) imposes a degree condition on all pairs of nonadjacent vertices (or on all vertices).

In [13] and [14], some sufficient conditions were described for a digraph to be Hamiltonian, in which a degree condition is required only for some pairs of nonadjacent vertices. Let us recall only the following theorem of them.

Theorem 1.1: (Bang-Jensen, Gutin, H.Li [13]). *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $min\{d(x), d(y)\} \geq n-1$ and $d(x) + d(y) \geq 2n-1$ for any pair of nonadjacent vertices x, y with a common in-neighbor. Then D is Hamiltonian.*

A cycle of a non-bipartite digraph D is called pre-Hamiltonian if it contains all the vertices of D except one. The concept of pre-Hamiltonian cycle for the balanced bipartite digraphs is the following:

A cycle of a balanced bipartite digraph D is called pre-Hamiltonian if it contains all the vertices of D except two.

A digraph D is called bipartite if there exists a partition X, Y of its vertex set into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$.

There are results analogous to the theorems of Ghouila-Houri, Nash-Williams, Woodall, Meyniel and Thomassen for balanced bipartite digraphs (see e.g., [15]) and the papers cited there.

Let x, y be a pair of distinct vertices in a digraph D. We call the pair $\{x, y\}$ dominating, if there is a vertex z in D such that $x \to z$ and $y \to z$.

An analogue of Theorem 1.1 for bipartite digraphs was given by R. Wang [16] and recently strengthened by the author[17].

Theorem 1.2: (R. Wang [16]). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 1$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ and $d(y) \geq a+1$ or $d(y) \geq 2a-1$ and $d(x) \geq a+1$. Then D is Hamiltonian.*

Let D be a balanced bipartite digraph of order $2a \geq 4$. For integer $k \geq 0$, we say that D satisfies condition B_k when $max\{d(x), d(y)\} \geq 2a - 2 + k$ for every pair of dominating vertices x and y.

Theorem 1.3: (Darbinyan [17]). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 4$. Suppose that D satisfies condition B_1, i.e., for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ or $d(y) \geq 2a-1$. Then either D is Hamiltonian or isomorphic to the digraph $D(8)$ (for the definition of $D(8)$, see Example 1).*

A balanced bipartite digraph of order $2m$ is even pancyclic if it contains a cycle of length $2k$ for any $2 \leq k \leq m$.

An even pancyclic version of Theorem 1.3 was proved in [18].

Theorem 1.4: (Darbinyan [18]). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ other than the directed cycle of length $2a$. If D satisfies condition B_1, i.e., $max\{d(x), d(y)\} \geq 2a-1$ for every dominating pair of vertices $\{x, y\}$, then either D contains cycles of all even lengths less than or equal to $2a$ or D is isomorphic to digraph $D(8)$.*

Theorem 1.5: (Darbinyan [18]). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$, which contains a pre-Hmiltonian cycle (i.e., a cycle of length $2a-2$). If D satisfies condition B_0, i.e., $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$, then for any k, $1 \leq k \leq a-1$, D contains a cycle of length $2k$ for every k, $1 \leq k \leq a-1$.*

In view of Theorem 1.5 it seems quite natural to ask whether a balanced bipartite digraph of order $2a$ in which $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$ contains a pre-Hamiltonian cycle (i.e., a cycle of length $2a-2$).

The underlying graph of a digraph D is the unique graph such that it contains an edge xy if $x \to y$ or $y \to x$ (or both).

In this paper we prove the following theorem.

Theorem 1.6: *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 10$ with partite sets X and Y. Assume that D satisfies condition B_0. Then either the underlying graph of D is 2-connected or D contains a cycle of length $2a-2$ unless D is isomorphic to the digraph $D(10)$ (for the definition of $D(10)$, see Example 2).*

2. Terminology and Notations

Terminology and notations not described below follow [1].

In this paper we consider finite digraphs without loops and multiple arcs. For a digraph D, we denote by $V(D)$ the vertex set of D and by $A(D)$ the set of arcs in D. The order of D is the number of its vertices. The arc of a digraph D directed from x to y is denoted by xy or $x \rightarrow y$. The notation $x \leftrightarrow y$ menas that $x \rightarrow y$ and $y \rightarrow x$ ($x \leftrightarrow y$ is called 2-cycle). We denote by $a(x, y)$ the number of arcs with end-vertices x and y. For disjoint subsets A and B of $V(D)$ we define $A(A \rightarrow B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \rightarrow B) \cup A(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we sometimes write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \rightarrow B$. The notation $A \leftrightarrow B$ means that $A \rightarrow B$ and $B \rightarrow A$. The out-neighbourhood of a vertex x is the set $N^+(x) = \{y \in V(D)/xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)/yx \in A(D)\}$ is the in-neighbourhood of x. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A/xy \in A(D)\}$ and $N^-(x, A) = \{y \in A/yx \in A(D)\}$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$ or $\langle A\rangle$ brevity. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. A cycle that contains all the vertices of D is a Hamiltonian cycle. A digraph D is strongly connected (or, just, strong) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y.

Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both). For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

A digraph D is called a bipartite digraph if there exists a partition X, Y of $V(D)$ into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$.

3. Examples

Example 1. Let $D(10)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4\}$ satisfying the following conditions: the induced subdigraph $\langle\{x_1, x_2, x_3, y_0, y_1\}\rangle$ is a complete bipartite digraph with partite sets $\{x_1, x_2, x_3\}$ and $\{y_0, y_1\}$; $\{x_1, x_2, x_3\} \rightarrow \{y_2, y_3, y_4\}$; $x_4 \leftrightarrow y_1$; $x_0 \leftrightarrow y_0$ and $x_i \leftrightarrow y_{i+1}$ for all $i \in [1, 3]$. $D(10)$ contains no other arcs.

It is not difficult to check that the digraph $D(10)$ is strongly connected and satisfies condition B_0, but the underlying graph of $D(10)$ is not 2-connected and $D(10)$ has no cycle of length 8. (It follows from the facts that $d(x_0) = d(x_4) = 2$ and x_0 (x_4) is on 2-cycle).
Example 2. Let $D(8)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$ satisfying the following conditions: the induced subdigraph $\langle\{x_1, x_2, y_0, y_1, y_3\}\rangle$ is a complete bipartite digraph with partite sets $\{x_1, x_2\}$ and $\{y_0, y_1, y_3\}$; $\{x_1, x_2, x_3\} \rightarrow \{y_2, y_3, y_4\}$; $x_3 \leftrightarrow y_3$; $x_0 \leftrightarrow y_0$ and $x_0 \leftrightarrow y_1$ and $D(8)$ contains no other arcs.

It is not difficult to check that the digraph $D(8)$ is strongly connected and satisfies condition B_0, but the underlying graph of $D(8)$ is not 2-connected and $D(8)$ has no cycle of length 6.

4. Proof of the Main Result

Proof of Theorem 1.6: Suppose, on the contrary, that the underlying graph of D is not 2-connected and D contains no cycle of length $2a-2$. Then $V(D) = A \cup B \cup \{u\}$, where A and B are nonempty disjoint subsets of vertices of D, the vertex u is not in $A \cup B$ and there are no arcs between A and B. Since D is strong, there are vertices $x \in A$ and $x_0 \in B$ such that $\{x, x_0\} \rightarrow u$, i.e., $\{x, x_0\}$ is a dominating pair. Note that x and x_0 belong to the same partite set, say X. Then $u \in Y$. By condition B_0 we have $max\{d(x), d(x_0)\} \geq 2a-2$. Without loss of generality, we assume that $d(x) \geq 2a-2$. From this and the fact that there are no arcs between A and B it follows that $a-2 \leq |Y \cap A| \leq a-1$.

Put $Y_1 := Y \cap A$. We will consider the cases $|Y_1| = a-2$ and $|Y_1| = a-1$ separately.

Case 1: $|Y_1| = a-2$.

Then $|Y \cap B| = 1$. Let $Y_1 := \{y_1, y_2, \dots, y_{a-2}\}$ and $Y \cap B := \{y_0\}$. It is not difficult to check that the vertex x and every vertex of $Y_1 \cup \{u\}$ form a 2-cycle, i.e., $x \leftrightarrow Y_1 \cup \{u\}$. Therefore, every pair of distinct vertices of $Y_1 \cup \{u\}$ is a dominating pair. This means that $Y_1 \cup \{u\}$ has at least $a-2$ vertices of degree at least $2a-2$ (maybe except, say y_{a-2}, or u). Then $d(y_1) \geq 2a-2$, since $a \geq 5$. From this it follows that $|X \cap A| = a-1$ and $X \cap B = \{x_0\}$ since there are no arcs between y_1 and B.

Put $X_1 := \{x_1, x_2, \dots, x_{a-1}\}$, where $x_1 = x$. Therefore, $B = \{x_0, y_0\}$. Since D is strong and since y_0 is not adjacent to any vertex of X_1, it follows that $y_0 \leftrightarrow x_0$, $u \rightarrow x_0$, $d(x_0) = 4$ and $d(y_0) = 2$. By condition B_0, we have $d(u) \geq 2a-2$ since $\{u, y_0\} \rightarrow x_0$.

Assume first that $d(y_i) \geq 2a-2$ for all $i \in [1, a-2]$. Then $Y_1 \leftrightarrow X_1$, since there are no arcs between Y_1 and $\{x_0\}$, i.e., the induced subdigraph $D\langle Y_1 \cap X_1 \rangle$ is a complete bipartite digraph with partite sets X_1 and Y_1. Since $d(u) \geq 2a-2$, it follows that the vertex u and at least $a-2$ vertices of X_1 form a 2 cycle. Now we can choose a vertex of X_1 other than x, say x_2, such that $u \leftrightarrow x_2$. Therefore, $x_1 u x_2 y_2 x_3 \dots x_{a-2} y_{a-2} x_{a-1} y_1 x_1$ is a cycle of length $2a-2$, which contradicts the supposition that D contains no cycle of length $2a-2$.

Assume second that Y_1 has a vertex, say y_{a-2}, of degree at most $2a-3$. Then from condition B_0 it follows that $d(y_i) \geq 2a-2$ for all $i \in [1, a-3]$ since $x \leftrightarrow Y_1 \cup \{u\}$. This implies that the subdigraph $D\langle X_1 \cup \{y_1, y_2, \dots, y_{a-3}\} \rangle$ is a complete bipartite digraph with partite sets X_1 and $\{y_1, y_2, \dots, y_{a-3}\}$. In particular, $y_1 \leftrightarrow X_1$. Then every pair of distinct vertices of X_1 is a dominating pair. Condition B_0 implies that X_1 has at least $a-2$ vertices, say $x_1, x_2, \dots, x_{a-2}$, of degree at least $2a-2$. Then

$$\{x_1, x_2, \dots, x_{a-2}\} \leftrightarrow Y_1 \cup \{u\},$$

in particular $y_{a-2} \leftrightarrow \{x_1, x_2, \dots, x_{a-2}\}$ and $u \leftrightarrow \{x_1, x_2, \dots, x_{a-2}\}$. Therefore, $y_1 x_{a-1} y_2 x_2 y_3 x_3 \dots y_{a-2}\ x_{a-2} u x_1 y_1$ is a cycle of length $2a-2$, which is a contradiction.

Case 2: $|Y_1| = a-1$.

Let now $Y_1 := \{y_1, y_2, \dots, y_{a-1}\}$. Then $Y \cap B = \emptyset$, i.e., $B \subseteq X$. Since D is strong, from condition B_0 it follows that $B = \{x_0\}$, $u \leftrightarrow x_0$ and $|X \cap A| = a-1$. Let now $X_1 := X \cap A = \{x_1, x_2, \dots, x_{a-1}\}$, where $x_1 = x$ (recall that $x_1 \rightarrow u$).

If $d(y_i) \geq 2a-2$ for all $i \in [1, a-1]$ then the subdigraph $D\langle X_1 \cup Y_1 \rangle$ is a complete bipartite digraph with partite sets X_1 and Y_1. Therefore, D contains a cycle of length $2a-2$, a contradiction.

Assume therefore that Y_1 has a vertex of degree at most $2a-3$. Observe that Y_1 may has at most three vertices of degree less than $2a-2$ since $d(x_1) \geq 2a-2$ (for otherwise Y_1 contains two vertices, say v and z, such that $\{v,z\} \rightarrow x_1$ and $max\{d(v), d(z)\} \leq 2a-3$, which contradicts condition B_0). We will consider the following four subcases depending on the number of vertices of Y_1, which have degree at most $2a-3$.

Subcase 2.1: Y_1 has exactly one vertex of degree less than $2a-2$.

Assume, without loss of generality, that $d(y_{a-1}) \leq 2a-3$ and $d(y_i) \geq 2a-2$ for all $i \in [1, a-2]$. Then it is easy to see that the subdigraph $D\langle X_1 \cup Y_1 \setminus \{y_{a-1}\}\rangle$ is a complete bipartite digraph with partite sets X_1 and $Y_1 \setminus \{y_{a-1}\}$ since $d(x_0, Y_1) = 0$. From strong connectedness of D it follows that $d^+(u, X_1) \geq 1$. If $u \rightarrow x_i$ for some $i \in [2, a-1]$, then by symmetry between the vertices $x_2, x_3, \ldots, x_{a-1}$, we can assume that $u \rightarrow x_2$. Then it is easy to see that $ux_2y_2x_3 \ldots y_{a-2}x_{a-1}y_1x_1u$ is a cycle of length $2a-2$, which is a contradiction. Assume therefore that

$$d^+(u, \{x_2, x_3, \ldots x_{a-1}\}) = 0. \tag{1}$$

Then $u \rightarrow x_1$, $d^+(y_{a-1}) \geq 1$ and $d^-(y_{a-1}) \geq 1$, since D is strong. If there exist two distinct vertices of X_1, say x_1 and x_2, such that $x_1 \rightarrow y_{a-1}$ and $y_{a-1} \rightarrow x_2$, then the cycle $x_1y_{a-1}x_2y_2x_3 \ldots x_{a-2}y_{a-2}x_{a-1}y_1x_1$ is a cycle of length $2a-2$, a contradiction. Assume therefore that there are no two distinct vertices x_i and x_j of X_1 such that $x_i \rightarrow y_{a-1}$ and $y_{a-1} \rightarrow x_j$. Then $d^+(y_{a-1}) = d^-(y_{a-1}) = 1$ and $y_{a-1} \leftrightarrow x_i$ for some $i \in [1, a-1]$. If $i = 1$, i.e., $x_1 \leftrightarrow y_{a-1}$. Then $d(y_{a-1}) = 2$. Now using (1) and the fact that $d(u, \{x_0, x_1\}) = 4$, we obtain

$$d(u) = d(u, \{x_0, x_1\}) + d^-(u, \{x_2, x_3, \ldots x_{a-1}\}) \leq a+2 \leq 2a-3,$$

which contradicts condition B_0 since $\{u, y_{a-1}\} \rightarrow x_1$ and $a \geq 5$. Therefore, $i \in [2, a-1]$.

Assume, without loss of generality, that $y_{a-1} \leftrightarrow x_{a-1}$. Then $a(x_i, y_{a-1}) = 0$ for all $i \in [1, a-2]$, in particular, $a(x_2, y_{a-1}) = a(x_3, y_{a-1}) = 0$. This together with (1) implies that $max\{d(x_2), d(x_3)\} \leq 2a-3$, which contradicts condition B_0 since $\{x_2, x_3\} \rightarrow y_1$. The discussion of Subcase 2.1 is completed.

Subcase 2.2: Y_1 has exactly two vertices of degree less than $2a-2$.

Assume, without loss of generality, that $d(y_{a-2}) \leq 2a-3$, $d(y_{a-1}) \leq 2a-3$ and $d(y_i) \geq 2a-2$ for all $i \in [1, a-3]$. Then it is easy to see that the subdigraph $D\langle X_1 \cup Y_1 \setminus \{y_{a-2}, y_{a-1}\}\rangle$ is a complete bipartite digraph with partite sets X_1 and $Y_1 \setminus \{y_{a-2}, y_{a-1}\}$ since $d(x_0, Y_1) = 0$.

For the discussion of Subcase 2.2 it is convient first to prove the following Claims 1 and 2 below.

Claim 1: If $x_j \rightarrow y_{a-2}$ for some $j \in [2, a-1]$, then $d^+(y_{a-2}, \{x_1, x_2, \ldots, x_{a-1}\} \setminus \{x_j\}) = 0$.

Proof of Claim 1: Assume, without loss of generality, that $x_{a-1} \rightarrow y_{a-2}$, i.e., $j = a-1$. Suppose that the claim is not true, i.e., $y_{a-2} \rightarrow x_i$ for some $i \in [1, a-2]$. We will consider the cases $i = 1$ and $i \in [2, a-2]$ separately.

Case. $i = 1$, *i.e.,* $y_{a-2} \rightarrow x_1$.

First we show that

$$d^+(u, \{x_2, x_3, \ldots, x_{a-1}\}) = 0. \tag{2}$$

Proof of (2): Suppose that (2) is not true, i.e., there is a $k \in [2, a-1]$ such that $u \rightarrow x_k$. If $k \in [2, a-2]$, we may assume, without loss of generality, that $u \rightarrow x_2$. Then the cycle $x_{a-1}y_{a-2}x_1ux_2y_1x_3y_2 \ldots\ x_{a-2}y_{a-3}x_{a-1}$ is a cycle of length $2a-2$, contradiction. Assume therefore that $k = a-1$. Then

$$u \rightarrow x_{a-1} \quad \text{and} \quad d^+(u, \{x_2, x_3, \ldots, x_{a-2}\}) = 0. \tag{3}$$

If $y_{a-2} \to x_l$, for some $l \in [2, a-2]$ (say $y_{a-2} \to x_2$), then the cycle $x_{a-1}y_{a-2}x_2y_1x_3y_2 \dots x_{a-2}y_{a-3}x_1$
ux_{a-1} is a cycle of length $2a-2$, a contradiction. Assume therefore that

$$d^+(y_{a-2}, \{x_2, x_3, \dots, x_{a-2}\}) = 0. \tag{4}$$

If $x_l \to u$ for some $l \in [2, a-2]$ (say $x_2 \to u$), then the cycle $x_{a-1}y_{a-2}x_1y_2x_3y_3 \dots y_{a-3}x_{a-2}y_1x_2ux_{a-1}$ is a cycle of length $2a-2$, a contradiction. Assume therefore that

$$d^-(u, \{x_2, x_3, \dots, x_{a-2}\}) = 0.$$

Combining this together with (3) and (4), we obtain

$$d(u, \{x_2, x_3, \dots, x_{a-2}\}) = d^+(y_{a-2}, \{x_2, x_3, \dots, x_{a-2}\}) = 0.$$

Therefore, since $a \geq 5$, we have $d(x_2)$ and $d(x_3) \leq 2a-3$, which contradicts condition B_0 since $\{x_2, x_3\} \to y_1$. This contradiction proves (2).

Since D is strong, from (2) it follows that $u \to x_1$. Therefore, $\{u, y_{a-2}\} \to x_1$, i.e., $\{u, y_{a-2}\}$ is a dominating pair. This together with condition B_0 implies that $d(u) \geq 2a-2$ since $d(y_{a-2}) \leq 2a-3$ (by our assumption). Now using (2), we obtain

$$2a-2 \leq d(u) = d(u, \{x_0, x_1\}) + d^-(u, \{x_2, x_3, \dots, x_{a-1}\}) \leq 4 + a - 2 = a + 2.$$

Hence, $a \leq 4$, which contradicts that $a \geq 5$. The discussion of the case $i = 1$ is completed.

Case. $i \in [2, a-2]$, *i.e.,* $y_{a-2} \to x_i$ *and* $y_{a-2}x_1 \notin A(D)$.

Assume, without loss of generality, that $y_{a-2} \to x_2$, i.e., $i = 2$. Now we prove that

$$d^+(u, \{x_3, x_4, \dots, x_{a-1}\}) = 0. \tag{5}$$

Proof of (5): Suppose that (5) is not true, i.e., there is an $l \in [3, a-1]$ such that $u \to x_l$. If $l = a-1$, i.e., $u \to x_{a-1}$, then the cycle $x_{a-1}y_{a-2}x_2y_2x_3 \dots y_{a-3}x_{a-2}y_1x_1ux_{a-1}$ is a cycle of length $2a-2$. Assume therefore that $l \in [3, a-2]$. Without loss of generality, we may assume that $u \to x_3$. Then the cycle $x_1ux_3y_2x_4 \dots y_{a-4}x_{a-2}y_{a-3}\ x_{a-1}y_{a-2}x_2y_1x_1$ is a cycle of length $2a-2$. In both cases we have a cycle of length $2a-2$, which is a contradiction. Therefore, (5) is true.

From (5) and strongly connectedness of D it follows that $u \to x_1$ or $u \to x_2$.

Assume first that $u \to x_1$. It is not difficult to show that

$$d^-(u, \{x_2, x_3, \dots, x_{a-2}\}) = 0. \tag{6}$$

Indeed, if $x_2 \to u$, then the cycle $y_{a-2}x_2ux_1y_1x_3y_2x_4 \dots x_{a-2}y_{a-3}x_{a-1}y_{a-2}$ has length $2a-2$; if $x_j \to u$ and $j \in [3, a-2]$, then (we may assume that $j = 3$, i.e., $x_3 \to u$) the cycle $x_{a-1}y_{a-2}x_2y_1x_3ux_1y_2x_4 \dots\ y_{a-4}x_{a-2}y_{a-3}x_{a-1}$ has length $2a-2$. In both cases we have a contradiction. Therefore, the equality (6) is true.

If $u \to x_2$, then from $y_{a-2} \to x_2$, $d(y_{a-2}) \leq 2a-3$ and condition B_0 it follows that $d(u) \geq 2a-2$. On the other hand, using (5) and (6) we obtain

$$2a-2 \leq d(u) = d(u, \{x_0, x_1\}) + d^+(u, \{x_2, x_3, \dots, x_{a-1}\}) + d^-(u, \{x_2, x_3, \dots, x_{a-1}\}) \leq 6.$$

Therefore, $a \leq 4$, which contradicts that $a \geq 5$. Assume therefore that $ux_2 \notin A(D)$. Then by (5) and (6) we have

$$d^+(u, \{x_2, x_3, \dots, x_{a-1}\}) = d^-(u, \{x_2, x_3, \dots, x_{a-2}\}) = 0.$$

In particular, $a(x_j, u) = 0$ for all $j \in [2, a-2]$. Since $a \geq 5$ and since $\{x_2, x_3\} \rightarrow y_1$, it follows that $d(x_2) = 2a-2$ or $d(x_3) = 2a-2$. If $d(x_2) = 2a-2$, then $\{y_{a-2}, y_{a-1}\} \rightarrow x_2$, and if $d(x_3) = 2a-2$, then $\{y_{a-2}, y_{a-1}\} \rightarrow x_3$. In each case we have a contradiction to condition B_0.

Assume second that $ux_1 \notin A(D)$ and $u \rightarrow x_2$. Then by condition B_0 we have $d(u) \geq 2a-2$ since $\{u, y_{a-2}\} \rightarrow x_2$ and $d(y_{a-2}) \leq 2a-3$. Now using (5), we obtain

$$2a-2 \leq d(u) = d(u, \{x_0, x_1\}) + d^+(u, \{x_2, x_3, \dots, x_{a-1}\}) + d^-(u, \{x_2, x_3, \dots, x_{a-1}\}) \leq a+2,$$

which is a contradiction, because of $a \geq 5$. Claim 1 is proved. □

Claim 2: If $x_j \rightarrow y_{a-2}$ for some $j \in [2, a-1]$, then $d^-(y_{a-2}, \{x_2, x_3, \dots, x_{a-1}\} \setminus \{x_j\}) = 0$.

Proof of Claim 2: Assume, without loss of generality, that $x_{a-1} \rightarrow y_{a-2}$, i.e., $j = a-1$. Suppose that the claim is not true, i.e., $x_l \rightarrow y_{a-2}$ for some $l \in [2, a-2]$. From Claim 1 and strongly connectedness of D it follows that $y_{a-2} \rightarrow x_{a-1}$. This together with condition B_0 and $max\{d(y_{a-2}), d(y_{a-1})\} \leq 2a-3$ implies that $y_{a-1}x_{a-1} \notin A(D)$.

Assume, without loss of generality, that $x_2 \rightarrow y_{a-2}$, i.e., $l = 2$. If $u \rightarrow x_2$, then the cycle $y_{a-2}x_{a-1}y_2x_3y_3 \dots x_{a-3}y_{a-3}x_{a-2}y_1x_1ux_2y_{a-2}$ has length $2a-2$, which is a contradiction. Let $u \rightarrow x_k$, where $k \in [3, a-2]$. We may assume that $k = 3$, i.e., $u \rightarrow x_3$. Then $y_{a-2}x_{a-1}y_1x_1ux_3y_2x_4y_3 \dots x_{a-2}y_{a-3}\, x_2y_{a-2}$ is a cycle of length $2a-2$, which is a contradiction. Therefore, we may assume that

$$d^+(u, \{x_2, x_3, \dots, x_{a-2}\}) = 0. \tag{7}$$

From (7) and strongly connectedness of D it follows that $u \rightarrow x_1$ or $u \rightarrow x_{a-1}$.

Assume first that $u \rightarrow x_1$. It is not difficult to see that if for some $j \in [3, a-2]$, say $j = 3$, $x_j \rightarrow u$, then the cycle $y_{a-2}x_{a-1}y_1x_3ux_1y_3x_4 \dots y_{a-3}x_{a-2}y_2x_2y_{a-2}$ has length $2a_2$, and if $x_{a-1} \rightarrow u$, then the cycle $y_{a-2}x_{a-1}ux_1y_1x_3y_3 \dots x_{a-3}y_{a-3}x_{a-2}y_2x_2y_{a-2}$ has length $2a-2$, which is a contradiction. Assume therefore that

$$d^-(u, \{x_3, x_4, \dots, x_{a-1}\}) = 0. \tag{8}$$

Now using (7) and (8), we obtain $a(u, x_j) = 0$ for all $j \in [3, a-2]$ and

$$d(u) = d(u, \{x_0, x_1\}) + d^+(u, \{x_2, x_3, \dots, x_{a-1}\}) + d^-(u, \{x_2, x_3, \dots, x_{a-1}\}) \leq 6 \leq 2a-3.$$

From (7), (8) and Claim 1 it follows that $d(x_j) \leq 2a-3$ for all $j \in [3, a-2]$. Hence, $a-2 = 3$, i.e., $a = 5$ and $d(x_3) \leq 2a-3$, and $d(x_2), d(x_4) \geq 2a-2$ since $\{x_2, x_3, \dots, x_{a-1}\} \rightarrow y_1$. From $y_{a-1}x_{a-1} \notin A(D)$ and $x_{a-1}u \notin A(D)$ $(a-1 = 4)$ it follows that $u \rightarrow x_{a-1}$, which is a contradiction since $\{u, y_{a-2}\} \rightarrow x_{a-1}$ and $max\{d(u), d(y_{a-2}\} \leq 2a-3$.

Assume second that $u \rightarrow x_{a-1}$ and $ux_1 \notin A(D)$. Since $\{u, y_{a-2}\} \rightarrow x_{a-1}$ and since $d(y_{a-2}) \leq 2a-3$ it follows that $d(u) \geq 2a-2$. On the other hand, using (7) and $ux_1 \notin A(D)$, we obtain

$$2a-2 \leq d(u) = d(u, \{x_0, x_1\}) + d^+(u, \{x_2, x_3, \dots, x_{a-1}\}) + d^-(u, \{x_2, x_3, \dots, x_{a-1}\}) \leq a+2,$$

which contradicts that $a \geq 5$. Claim 2 is proved. $\square$

Now we are ready to complete the discussion of Subcase 2.2.

Assume that $d^-(y_j, \{x_2, x_3, \ldots, x_{a-1}\}) \neq 0$ for $j = a-2$ or $a-1$ (say $j = a-2$). Assume, without loss of generality, that $x_{a-1} \to y_{a-2}$. From Claims 1 and 2 it follows that

$$d^+(y_{a-2}, \{x_1, x_2, x_3, \ldots, x_{a-2}\}) = d^-(y_{a-2}, \{x_2, x_3, \ldots, x_{a-2}\}) = 0. \tag{9}$$

Therefore, $d(x_i) \leq 2a-2$ for all $i \in [2, a-2]$ since $a(x_i, y_{a-2}) = 0$. From strongly connectedness of D and (9) it follows that $y_{a-2} \to x_{a-1}$. This together with $max\{d(y_{a-2}), d(y_{a-1})\} \leq 2a-3$ and condition B_0 implies that $y_{a-1}x_{a-1} \notin A(D)$. Therefore,

$$d^+(y_{a-1}, \{x_1, x_2, x_3, \ldots, x_{a-2}\}) \neq 0$$

since D is strong. Now we apply Claim 1 to y_{a-1} we conclude that $x_{a-1}y_{a-1} \notin A(D)$. Then $a(x_{a-1}, y_{a-1}) = 0$ and $d(x_{a-1}) \leq 2a-2$. Since $\{x_2, x_3, \ldots, x_{a-1}\} \to y_1$, from condition B_0 it follows that $\{x_2, x_3, \ldots, x_{a-1}\}$ has at least $a-3$ vertices of degree at least $2a-2$. In particular, $d(x_2) \geq 2a-2$ or $d(x_3) \geq 2a-2$. Without loss of generality, we assume that $d(x_2) \geq 2a-2$. Then $x_2 \to \{u, y_{a-1}\}$ since $a(x_2, y_{a-2}) = 0$. Now using (9) with respect to y_{a-1}, we obtain

$$d^+(y_{a-1}, \{x_1, x_3, x_4, \ldots, x_{a-1}\}) = d^-(y_{a-1}, \{x_3, x_4, \ldots, x_{a-1}\}) = 0. \tag{10}$$

In particular, from (9) and (10) we have $d^-(x_1, \{y_{a-2}, y_{a-1}\}) = 0$. Therefore, $x_1 \to y_{a-2}$ and $u \to x_1$ since $d(x_1) \geq 2a-2$. Hence, the cycle $x_2ux_1y_{a-2}x_{a-1}y_1x_3y_2x_4 \ldots y_{a-4}x_{a-2}y_{a-3}x_2$ is a cycle of length $2a-2$, which is a contradiction.

Assume now that

$$A(\{x_2, x_3, \ldots, x_{a-1}\} \to \{y_{a-2}, y_{a-1}\}) = 0.$$

Then, since D is strong, it follows that $x_1 \to \{y_{a-2}, y_{a-1}\}$. From the last equality we have $d(x_j) \leq 2a-2$ for all $j \in [2, a-1]$. This together with $\{x_2, x_3, \ldots, x_{a-1}\} \to y_1$ implies that $\{x_2, x_3, \ldots, x_{a-1}\}$ has at least $a-3$ vertices of degree equal to $2a-2$. Assume, without loss of generality, that $d(x_2) = 2a-2$. Then $\{y_{a-2}, y_{a-1}\} \to x_2$, which is a contradiction since $d(y_{a-2}) \leq 2a-3$ and $d(y_{a-1} \leq 2a-3$. In each case we obtain a contradiction, and hence, the discussion of Subcase 2.2 is completed.

Subcase 2.3: Y_1 has exactly three vertices of degree less than $2a-2$.

Assume, without loss of generality, that $d(y_j) \leq 2a-3$ for all $j \in [a-3, a-1]$ and $d(y_i) \geq 2a-2$ for all $i \in [1, a-4]$. Then it is easy to see that the subdigraph $D\langle\{X_1 \cup \{y_1, y_2, \ldots, y_{a-4}\}\rangle$ is a complete bipartite digraph and $d^-(x_i, \{y_{a-3}, y_{a-2}, y_{a-1}\}) \leq 1$ for all $i \in [1, a-1]$. This together with condition B_0 implies that $\{x_2, x_3, \ldots, x_{a-1}\}$ has at least $a-3$ vertices of , say $x_2, x_3, \ldots, x_{a-2}$, of degree equal to $2a-2$. Then $x_1 \leftrightarrow u$, $x_i \to \{y_{a-3}, y_{a-2}, y_{a-1}\}$ if $i \in [1, a-2]$, and $x_j \leftrightarrow u$ if $j \in [2, a-2]$. Now it is not difficult to see that for every $i \in [1, a-2]$ there is a $j \in [a-3, a-1]$ such that $x_i \leftrightarrow y_j$. Because of the symmetry between the vertices $x_1, x_2, \ldots, x_{a-2}$, we can assume, $x_1 \leftrightarrow y_{a-3}$.

Assume first that

$$A(\{y_{a-2}, y_{a-1}\} \to \{x_4, x_5, \ldots, x_{a-1}\}) \neq \emptyset.$$

Let $y_{a-2} \to x_{a-1}$. Then the cycle $x_2ux_3y_{a-3}\ x_1y_{a-2}x_{a-1}y_1x_4y_2\dots x_{a-2}y_{a-4}x_2$ has length $2a-2$, if $a \geq 6$, and the cycle $x_2ux_3y_2x_1y_3x_4y_1x_2$, if $a=5$ has length $2a-2$, which is a contradiction.

Assume second that

$$A(\{y_{a-2}, y_{a-1}\} \to \{x_4, x_5, \dots, x_{a-1}\}) = \emptyset.$$

From $x_1 \leftrightarrow y_{a-3}$, $max\{d(y_{a-3}), d(y_{a-2}), d(y_{a-1})\} \leq 2a-3$ and condition B_0 it follows that

$$d^-(x_1, \{y_{a-2}, y_{a-3}\}) = 0 \quad \text{and} \quad min\{d^+(y_{a-2}, \{x_2, x_3\}), d^+(y_{a-1}, \{x_2, x_3\}\}) \geq 1. \qquad (11)$$

Without loss of generality, we assume that $y_{a-2} \to x_2$. If $a \geq 6$, then the cycle $y_{a-2}x_2y_{a-3}x_1ux_3y_1x_{a-1}\ y_2x_4y_3\dots x_{a-3}y_{a-4}x_{a-2}y_{a-2}$ is a cycle of length $2a-2$, which is a contradiction. Assume therefore that $a=5$. Now using (11), $y_3 \to x_2$, $y_2 \to x_1$ and condition B_0, we obtain $y_3 \to x_3$ and $d^+(y_4, \{x_2, x_3\}) = 0$. Thus, we have that $D\langle\{x_1, x_2, x_3, u, y_1\}\rangle$ is a complete bipartite digraph with partite sets $\{x_1, x_2, x_3\}$ and $\{u, y_1\}$, $\{x_1, x_2, x_3\} \to \{y_2, y_3, y_4\}$, $x_4 \leftrightarrow y_1$, $x_i \leftrightarrow y_{i+1}$ for all $i \in [1, 3]$ and $x_0 \leftrightarrow u$. It is easy to check that the obtained digraph is strongly connected and isomorphic to $D(10)$, which satisfies condition B_0, but has no cycle of length 8. The theorem is proved. □

From Theorems 1.5 and 1.6 follows the following corollary follows.

Corollary: *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 10$. Assume that $d(x)+d(y) \geq 4a-3$ for every dominaiting pair of vertices x and y. Then either the underlying graph of D is 2-connected or D contains a cycle of length k for every $k \in [1, a-1]$ unless D is isomorphic to the digraph $D(10)$.*

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Հավասարակշռված երկմաս կողմնորոշված գրաֆների նախահամիլտոնյան ցիկլերի մասին

Ս. Դարբինյան

Ամփոփում

Հավասարակշռված երկմաս կողմնորոշված գրաֆի կողմնորոշված ցիկլը կոչվում է նախահամիլտոնյան, եթե այն պարունակում է այդ գրաֆի բոլոր գագաթները ‘ բացի երկուսից:

Ներկա աշխատանքում ցույց է տրվում հետևյալ պնդումը.

Թեորեմ: Դիցուք՝ D-ն $2a \geq 10$ գագաթանի հավասարակշռված երկմաս կողմնորոշված գրաֆ է: Եթե այդ գրաֆի գագաթների ցանկացած հաղթող զույգի առնվազն մեկ գագաթի լոկալ աստիճանը փոքր չէ $2a-2$ թվից, ապա D-ն պարունակում է նախահամիլտոնյան ցիկլ կամ D-ի չկողմնորոշված հիմք գրաֆը 2-կապակցված է կամ D-ն իզոմորֆ է մեկ 10 գագաթանի գրաֆին:

О предгамильтоновых контуров в сбалансированных двудольных орграфах

С. Дарбинян

Аннотация

Ориентированный контур проходящий через все вершины сбалансированного двудольного орграфа, кроме двух вершин, называется предгамильтоновым контуром. В настоящей статье доказывается:

Теорема: Пусть D - $2a$-вершинный $(a \geq 5)$ сбалансированный двудольный орграф. Если для любых доминирующих пар вершин по крайней мере одна вершина имеет локальную степень не меньше чем $2a-2$, то D содержит предгамильтоновый контур или неориентированная основа граф орграфа является D 2-связной или D изоморфен одному орграфу с десятью вершинами.