Mathematical Problems of Computer Science 47, 30–36, 2017.

On Longest Cycles in 2-connected Graphs

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Abstract

For a graph G, n denotes the order (the number of vertices) of G, c the order of a longest cycle in G (called circumference), p the order of a longest path and δ the minimum degree. In 1952, Dirac proved: (i) if G is a 2-connected graph, then $c \geq \min\{n, 2\delta\}$. The bound 2δ in (i) was enlarged independently by Bondy (1971), Bermond (1976) and Linial (1976) in terms of σ_2 - the minimum degree sum of two nonadjacent vertices: (ii) if G is a 2-connected graph, then $c \geq \min\{n, \sigma_2\}$. In this paper two further extensions of (i) and (ii) are presented by incorporating p and the length of a vine on a longest path of G as new parameters along with n, δ and σ_2.

Keywords: Hamilton cycle, Dominating cycle, Longest cycle, Longest path, Minimum degree, Degree sums.

1. Introduction

We consider only finite undirected graphs with neither loops nor multiple edges. A good reference for any undefined terms is [2].

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. Let n be the order (the number of vertices) of G, c the order of a longest cycle (called circumference) in G and p the order of a longest path. The minimum degree sum of two nonadjacent vertices in G is denoted by σ_2. In particular, the minimum degree σ_1 is denoted by δ. We use $N(v)$ to denote the set of all neighbors of vertex v and $d(v) = |N(v)|$ to denote the degree of vertex v. A graph G is hamiltonian if G contains a Hamilton cycle, that is a simple spanning cycle. A cycle C of G is called a dominating cycle if every edge of G has at least one of its end vertices on C, or, equivalently, if $G - V(C)$ contains no edges.

The earliest nontrivial lower bound for the circumference was obtained in 1952 due to Dirac [4] in terms of δ and n.

Theorem A: [4]. *In every 2-connected graph,* $c \geq \min\{n, 2\delta\}$.

The bound 2δ in Theorem A was enlarged independently by Bondy [1], Bermond [3] and Linial [5] in terms of σ_2.

Theorem B: [1],[3],[5]. *In every 2-connected graph,* $c \geq \min\{n, \sigma_2\}$.

In this paper two further extensions of these results are presented by incorporating p and the length of a vine on a longest path of G in corresponding bounds as new parameters along with n, δ and σ_2. The vine's definition needs some additional notation.

If Q is a path or a cycle, then the length of Q, denoted by $l(Q)$, is $|E(Q)|$ - the number of edges in Q. We write a cycle Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. For $x \in V(Q)$, we denote the successor and the predecessor of x on $\overrightarrow{Q}$ (if such vertices exist) by x^+ and x^-, respectively. We use $P = x\overrightarrow{P}y$ to denote a path with end vertices x and y in the direction from x to y. We say that vertex z_1 precedes vertex z_2 on a path $\overrightarrow{Q}$ if z_1, z_2 occur on $\overrightarrow{Q}$ in this order, and indicate this relationship by $z_1 \prec z_2$. We will write $z_1 \preceq z_2$ when either $z_1 = z_2$ or $z_1 \prec z_2$.

Let $P = x\overrightarrow{P}y$ be a path. A vine of length m on P is a set

$$\{L_i = x_i\overrightarrow{L}_i y_i : 1 \leq i \leq m\}$$

of internally-disjoint paths such that

(a) $V(L_i) \cap V(P) = \{x_i, y_i\} \quad (i = 1, ..., m)$,
(b) $x = x_1 \prec x_2 \prec y_1 \preceq x_3 \prec y_2 \preceq x_4 \prec ... \preceq x_m \prec y_{m-1} \prec y_m = y$ on P.

The following result guarantees the existence of at least one vine in a 2-connected graph.

Lemma: (The Vine Lemma) [4]. *Let G be a k-connected graph and P a path in G. Then there are $k-1$ pairwise-disjoint vines on P.*

In the paper, we obtain a lower bound for the circumference in terms of n, σ_2 and the length m of a vine on a longest path of G.

Theorem 1: *Let G be a 2-connected graph and $\{L_1, L_2, ..., L_m\}$ be a vine on a longest path of G. Then*

$$c \geq \min\{n, \sigma_2 + m - 2\}.$$

The minimum degree version of Theorem 1 follows immediately.

Corollary 1: *Let G be a 2-connected graph and $\{L_1, L_2, ..., L_m\}$ be a vine on a longest path of G. Then*

$$c \geq \min\{n, 2\delta + m - 2\}.$$

If $m = 1$ in Theorem 1, then clearly G is hamiltonian. Therefore, Theorem 1 is an extension of Theorems A and B by incorporating parameter m along with n and σ_2.

Next, we obtain a lower bound for the circumference c in terms of σ_2 and p.

Theorem 2: *Let G be a 2-connected graph. Then*

$$c \geq \begin{cases} p & \text{when} \quad p \leq \sigma_2, \\ p-1 & \text{when} \quad \sigma_2 + 1 \leq p \leq \sigma_3 - 2, \\ \sqrt{2p - 10 + \frac{1}{4}(\sigma_2 - 7)^2} + \frac{1}{2}(\sigma_2 + 1) & \text{when} \quad p \geq \sigma_3 - 1. \end{cases}$$

Theorem 2 can be considered as another extension of Theorems A and B. Indeed, if $p \leq \sigma_2$, then by Theorem 2, $c \geq p$, implying that $c = p = n \geq \min\{n, \sigma_2\}$. Next, if $\sigma_2 + 1 \leq p \leq \sigma_3 - 2$, then by Theorem 2, $c \geq p - 1 \geq \sigma_2 \geq \min\{n, \sigma_2\}$. Finally, if $p \geq \sigma_3 - 1$, then observing that $2\sigma_3 \geq 3\sigma_2$, we get

$$\sqrt{2p - 10 + \frac{1}{4}(\sigma_2 - 7)^2} \geq \sqrt{2(\sigma_3 - 1) - 10 + \frac{1}{4}(\sigma_2 - 7)^2} = \frac{1}{2}(\sigma_2 - 1),$$

and by Theorem 2, $c \geq \sigma_2 \geq \min\{n, \sigma_2\}$.

The minimum degree version of Theorem 2 follows immediately.
Corollary 2: *Let G be a 2-connected graph. Then*

$$c \geq \begin{cases} p & \text{when} \quad p \leq 2\delta, \\ p-1 & \text{when} \quad 2\delta+1 \leq p \leq 3\delta-2, \\ \sqrt{2p-10+(\delta-\frac{7}{2})^2}+\delta+\frac{1}{2} & \text{when} \quad p \geq 3\delta-1. \end{cases}$$

The special cases $c \geq p$ and $c \geq p-1$ in Theorem 2 can be interpreted in terms of Hamilton and dominating cycles by the following two propositions.
Proposition 1: [6]. *A connected graph is hamiltonian if and only if $c = p$.*
Proposition 2: [6]. *Let G be a connected graph with $c \geq p-1$. Then every longest cycle in G is a dominating cycle.*

To show that the bounds in Corollary 2 (as well as in Theorem 2) are sharp, observe first that in general, $p \geq c$, that is $c = p$ when $p \leq 2\delta$, implying that the bound $c \geq p$ in Corollary 2 cannot be replaced by $c \geq p+1$. On the other hand, the graph $K_{\delta,\delta+1}$ with $p = 2\delta+1$ and $c = 2\delta = p-1$ shows that the condition $p \leq 2\delta$ cannot be relaxed to $p \leq 2\delta+1$. In addition, the graph $K_{\delta,\delta+1}$ with $c = p$ shows that the bound $c \geq p-1$ (when $2\delta+1 \leq p \leq 3\delta-2$) cannot be replaced by $c \geq p$. Further, the graph $K_2 + 3K_{\delta-1}$ with $n = p = 3\delta-1$ and $c = 2\delta \leq p-2$ shows that the condition $p \leq 3\delta-2$ cannot be relaxed to $p \leq 3\delta-1$. Finally, the same graph $K_2 + 3K_{\delta-1}$ with $p = 3\delta-1$ and

$$c = 2\delta = \sqrt{2p-10+\left(\delta-\frac{7}{2}\right)}+\delta+\frac{1}{-}2,$$

shows that the bound $\sqrt{2p-10+(\delta-\frac{7}{2})}+\delta+\frac{1}{2}$ in Corollary 2 cannot be improved to $\sqrt{2p-10+(\delta-\frac{7}{2})}+\delta+1$.
The following theorem will be useful.
Theorem C: [6]. *Let G be a 2-connected graph. Then either (i) $c \geq p-1$ or (ii) $c \geq \sigma_3-3$ or (iii) $\kappa = 2$ and $p \geq \sigma_3-1$.*

2. Preliminaries

The following lemma can be proved by standard arguments (called Dirac and Ore arguments).
Lemma 1: *Let G be a connected graph and $P = x\overrightarrow{P}y$ a longest path in G.*
(i) If $xz, yz^- \in E(G)$ for some $z \in V(x^+\overrightarrow{P}y)$, then $c = p = n$, that is G is hamiltonian.
(ii) If $d(x)+d(y) \geq p$, then $c = p = n$.
(iii) Let $z_1, z_2 \in V(P)$ and $z_1 \prec z_2$. If $xz, yz \notin E(G)$ for each $z \in V(z_1^+\overrightarrow{P}z_2^-)$, then either $c = p$ or $p \geq d(x)+d(y)-2+|z_1\overrightarrow{P}z_2|$.

The next lemma is crucial for the proof of Theorems 1 and 2.
Lemma 2: *Let G be a 2-connected graph and $\{L_1, L_2, ..., L_m\}$ be a vine on a longest path of G. Then*

$$c \geq \frac{2p-10}{m+1}+4.$$

3. Proofs

Proof of Lemma 2: *Let $P = x\overrightarrow{P}y$ be a longest path in G. Put*

$$L_i = x_i\overrightarrow{L}_i y_i \ \ (i = 1, ..., m), \ \ A_1 = x_1\overrightarrow{P}x_2, \ \ A_m = y_{m-1}\overrightarrow{P}y_m,$$

$$A_i = y_{i-1}\overrightarrow{P}x_{i+1} \ \ (i = 2, 3, ..., m-1), \ \ B_i = x_{i+1}\overrightarrow{P}y_i \ \ (i = 1, ..., m-1),$$

$$|A_i| - 1 = a_i \ \ (i = 1, ..., m), \ \ |B_i| - 1 = b_i \ \ (i = 1, ..., m-1).$$

By combining appropriate L_i, A_i, B_i, we form $m + 1$ different cycles to obtain a lower bound for the circumference as the mean of their orders.

$$Q_1 = \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{m} L_i,$$

$$Q_2 = \bigcup_{i=1}^{m-1} A_i \cup B_{m-1} \cup \bigcup_{i=1}^{m-1} L_i,$$

$$Q_3 = \bigcup_{i=2}^{m} A_i \cup B_1 \cup \bigcup_{i=2}^{m} L_i,$$

$$R_i = B_i \cup A_{i+1} \cup B_{i+1} \cup L_{i+1} \ \ (i = 1, ..., m-2).$$

Since $|L_i| \geq 2$ $(i = 1, ..., m)$, we have

$$c \geq |Q_1| = \sum_{i=1}^{m} a_i + \sum_{i=1}^{m}(|L_i| - 1) \geq \sum_{i=1}^{m} a_i + m,$$

$$c \geq |Q_2| = b_{m-1} + \sum_{i=1}^{m-1} a_i + \sum_{i=1}^{m-1}(|L_i| - 1) \geq b_{m-1} + \sum_{i=1}^{m-1} a_i + m - 1,$$

$$c \geq |Q_3| = b_1 + \sum_{i=2}^{m} a_i + \sum_{i=2}^{m}(|L_i| - 1) \geq b_1 + \sum_{i=2}^{m} a_i + m - 1,$$

$$c \geq |R_i| = b_i + a_{i+1} + b_{i+1} + |L_{i+1}| - 1$$

$$\geq b_i + a_{i+1} + b_{i+1} + 1 \ \ (i = 1, ..., m-2).$$

By summing, we get

$$(m+1)c \geq (2\sum_{i=1}^{m} a_i + 2\sum_{i=1}^{m-1} b_i) + 2\sum_{i=2}^{m-1} a_i + 4m - 4$$

$$\geq 2(\sum_{i=1}^{m} a_i + \sum_{i=1}^{m-1} b_i + 1) + 4m - 6 = 2p + 4m - 6,$$

implying that

$$c \geq \frac{2p - 10}{m + 1} + 4.$$

Lemma 2 is proved. ■

Proof of Theorem 1: If $m=1$, then $xy \in E(G)$ and by Lemma 1(i), $c=p$. Let $m \geq 2$. Put $L_i = x_i \overrightarrow{L}_i y_i$ $(i=1,...,m)$ and let

$$A_i,\ B_i,\ a_i,\ b_i,\ Q_i$$

be as defined in the proof of Lemma 2. We choose $L_1, L_2, ..., L_m$ so as to minimize m as well as b_1 and b_{m-1}.

Case 1: $m=2$.

It follows that $N(x) \cup N(y) \subseteq V(A_1 \cup A_2)$. By Lemma 1$(iii)$, either $c=p$ or $p = a_1 + a_2 + b_1 + 1 \geq d(x) + d(y) - 1 + b_1$, implying that

$$c \geq |Q_1| = a_1 + a_2 + 2 \geq d(x) + d(y) = d(x) + d(y) + m - 2.$$

Case 2: $m=3$.

Let $xz_1, yz_2 \in E(G)$ for some $z_1, z_2 \in V(P)$. If $z_2 \prec z_1$, then $\{xz_1, yz_2\}$ is a vine consisting of two paths (edges) and we can argue as in Case 1. By the choice of L_1, L_2, L_3,

$$N(x) \subseteq V(A_1 \cup A_2), \quad N(y) \subseteq V(A_2 \cup A_3)$$

and $z_1 \preceq z_2$ for each $z_1 \in N(x)$ and $z_2 \in N(y)$. Therefore, $a_1 + a_2 + a_3 \geq d(x) + d(x) - 2$ and

$$c \geq |Q_1| = a_1 + a_2 + a_3 + 3$$

$$\geq d(x) + d(x) + 1 = d(x) + d(x) + m - 2.$$

Case 3: $m \geq 4$.

By the choice of $L_1, L_2, ..., L_m$,

$$N(x) \subseteq V(A_1 \cup A_2), \quad N(y) \subseteq V(A_{m-1} \cup A_m)$$

and $z_1 \prec z_2$ for each $z_1 \in N(x)$ and $z_2 \in N(y)$. Observing also that

$$a_1 + a_2 \geq d(x) - 1, \quad a_{m-1} + a_m \geq d(y) - 1,$$

we get

$$c \geq |Q_1| = \sum_{i=1}^{m} a_i + m = (a_1 + a_2 + a_{m-1} + a_m) + \sum_{i=3}^{m-2} a_i + m$$

$$\geq d(x) + d(y) - 2 + \sum_{i=3}^{m-2} a_i + m \geq d(x) + d(y) + m - 2.$$

Theorem 1 is proved. ■

Proof of Theorem 2: Let $P = x\overrightarrow{P}y$ be a longest path in G.

Case 1: $p \leq \sigma_2$.

If $xy \in E(G)$, then by Lemma 1(i), $c=p$. Let $xy \notin E(G)$. Then $d(x) + d(y) \geq \sigma_2 \geq p$ and by Lemma 1(ii), again $c=p$.

Case 2: $\sigma_2 + 1 \leq p \leq \sigma_3 - 2$.

If $c \geq \sigma_3 - 3$, then by the hypothesis, $c \geq p-1$. Next, if $\kappa = 2$ and $p \geq \sigma_3 - 1$, then $p \geq \sigma_3 - 1 \geq p+1$, a contradiction. Hence, by Theorem C, $c \geq p-1$.

Case 3: $p \geq \sigma_3 - 1$.

Since G is 2-connected, then by the Vine Lemma, there is a vine $\{L_1, ..., L_m\}$ on P. By Theorem 1, $m \leq c - d(x) - d(y) + 2 \leq c - \sigma_2 + 2$. Using Lemma 2, we get

$$c \geq \frac{2p - 10}{m + 1} + 4 \geq \frac{2p - 10}{c - \sigma_2 + 3} + 4,$$

implying that

$$c \geq \sqrt{2p - 10 + \frac{1}{4}(\sigma_2 - 7)^2} + \frac{1}{2}(\sigma_2 + 1).$$

Theorem 2 is proved. ■

References

[1] J.A. Bondy, "Large cycles in graphs", *Discrete Math.*, vol. 1, pp. 121-131, 1971.

[2] J. A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[3] J. C. Bermond, "On Hamiltonian walks", *Congr Numer*, vol. 15, pp. 41-51, 1976.

[4] G. A. Dirac, "Some theorems on abstract graphs", *Proc. London, Math. Soc.*, vol. 2, pp. 69-81, 1952.

[5] N. Linial, "A lower bound on the circumference of a graph", *Discrete Math.*, vol. 15, pp. 297-300, 1976.

[6] K. Ozeki and T. Yamashita, "Length of longest cycles in a graph whose relative length is at least two", *Graphs and Combin.*, vol. 28, pp. 859-868, 2012.

Submitted 22.10.2016, accepted 17.01.2017.

2-կապակցված գրաֆների ամենաերկար ցիկլերի մասին

Մ. Քուլաքզյան և Ժ. Նիկողոսյան

Ամփոփում

Դիցուք n-ը նշանակում է G գրաֆի գագաթների քանակը, c-ն G-ի՝ ամենաերկար ցիկլի երկարությունը, p-ն՝ ամենաերկար շղթայի գագաթների քանակը և δ-ն՝ գրաֆի նվազագույն աստիճանը: 1952-ին Դիրակը ապացուցեց, որ (i) եթե G-ն 2-կապակցված գրաֆ է, ապա $c \geq \min\{n, 2\delta\}$: Դիրակի 2δ ներքին գնահատականը իրարից անկախ ընդլայնեցին Բոնդին (1971), Բերմոնդը և Լինիալը (1976), օգտագործելով σ_2 (ոչ հարևան երկու գագաթների աստիճանների նվազագույն գումարը) պարամետրը, (ii) եթե G-ն 2-կապակցված գրաֆ է, ապա $c \geq \min\{n, \sigma_2\}$: Ներկա աշխատանքում (i) և (ii) ընդլայնումները ավելի են ընդլայնվում՝ գնահատականների մեջ ներմուծելով p-ն և G գրաֆի ամենաերկար շղթայի բաղեղի երկարությունը որպես նոր պարամետրեր n, δ, σ_2 պարամետրերի կողքին:

О длиннейших циклах 2-связных графов

М. Кулакзян и Ж. Никогосян

Аннотация

Пусть n, c, p и δ обозначают число вершин графа G, длина длиннейшего цикла, число вершин длиннейшей цепи и минимальная степень графа. В 1952 году Дирак доказал, что (i) если G является 2-связным графом, то $c \geq \min\{n, 2\delta\}$: Эту оценку независимо расширили Бонди (1971), Бермонд и Линиал (1976) с помощью параметра σ_2 (минимальная сумма степеней двух не соседних вершин): (ii) если G является 2-связным графом, то $c \geq \min\{n, \sigma_2\}$. В настоящей работе представлены две новые расширения оценок (i) и (ii) помощью параметров p и длины плюща длиннейшей цепи графа G на ряду с параметрами n, δ, σ_2.