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# On Long Cycles in Graphs in Terms of Degree Sequences

Mossine S. Koulakzian

Institute for Informatics and Automation Problems of NAS RA
e-mail: mossine@hotmail.com

**Abstract**

Let $G$ be a graph on $n$ vertices with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. Let $m$ be the number of connected components of $G$, $c$ the circumference - the order of a longest cycle, $p$ the order of a longest path in $G$ and $\sigma_s$ the minimum degree sum of an independent set of $s$ vertices. In this paper it is shown that in every graph $G$, $c \geq d_{\sigma_m+m} + 1$. This bound is best possible and generalizes the earliest lower bound for the circumference due to Dirac (1952): $c \geq \delta + 1 = d_1 + 1$. As corollaries, we have: (i) $c \geq d_{\delta+1} + 1$; (ii) if $d_{\sigma_m+m} \geq p - 1$, then $c = p$; (iii) if $G$ is a connected graph with $d_{\delta+1} \geq p - 1$, then $G$ is hamiltonian; (iv) if $d_{\sigma_m+m} \geq n - 1$, then $G$ is hamiltonian.

**Keywords:** Hamilton cycle, Longest cycle, Circumference, Minimum degree, Degree sequence.

## 1. Introduction

We consider only finite undirected graphs with neither loops nor multiple edges. A good reference for any undefined terms is [1].

The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges by $E(G)$. Let $n$ be the order (the number of vertices) of $G$, $c$ the order of a longest cycle (called circumference) in $G$ and $p$ the order of a longest path. The minimum degree in $G$ is denoted by $\delta$ and the minimum degree sum of an independent set of $s$ vertices will be denoted by $\sigma_s$. Let $d_1, d_2, ..., d_n$ be the degree sequence in $G$ with $\delta = d_1 \leq d_2 \leq ... \leq d_n$. We use $N(v)$ to denote the set of all neighbors of vertex $v$ and $d(v) = |N(v)|$ to denote the degree of vertex $v$.

A path (simple path) of order $m$ is a sequence of distinct vertices $v_1, ..., v_m$, denoted by $v_1v_2...v_m$, such that $v_{i-1}v_i$ is an edge for all $2 \leq i \leq m$. Similarly. a cycle of order $m$ is a sequence of distinct vertices $v_1, ..., v_m$, denoted by $v_1v_2...v_mv_1$, such that $v_{i-1}v_i$ and $v_mv_1$ are edges for all $2 \leq i \leq m$. In particular, for $m = 2$, $v_1v_2v_1$ is a cycle of order 2; and for $m = 1$, $v_1v_1$ is a cycle of order 1. So, by the definition, every vertex (edge) can be considered as a cycle of order 1 (2, respectively).A graph $G$ is hamiltonian if $G$ contains a Hamilton cycle, that is a simple spanning cycle.

We write a cycle (a path) $Q$ with a given orientation by $\overrightarrow{Q}$. The reverse sequence of vertices of $\overrightarrow{Q}$ is denoted by $\overleftarrow{Q}$. For $x \in V(Q)$, we denote the predecessor of $x$ on $\overrightarrow{Q}$ (if such vertices exist) by $x^-$. We use $P = x\overrightarrow{P}y$ to denote a path with end vertices $x$ and $y$ in the direction from $x$ to $y$.

The earliest and simplest lower bound for the circumference was obtained in 1952 due to Dirac [2] in terms of minimum degree $\delta$.

**Theorem A:** [2]. *In every graph,* $c \geq \delta + 1$.

Theorem A is best possible. However, the bound $c \geq \delta + 1$ in Theorem A is equivalent to $c \geq d_1 + 1$, which is far from being best possible. In this paper we present an improvement of Theorem A in terms of degree sequences without any additional conditions.

**Theorem 1:** *Let $G$ be a graph with $m$ connected components and degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. Then $c \geq d_{\sigma_m+m} + 1$.*

If $G = mK_{\delta+1}$, then $c = \delta + 1 = d_{m\delta+m} + 1 = d_{\sigma_m+m} + 1$. This graph example shows that the bound $d_{\sigma_m+m} + 1$ in Theorem 1 cannot be replaced by $d_{\sigma_m+m} + 2$. Next, let $G = m(K_\delta + \overline{K}_{\delta+1})$. Then $d_{\sigma_m+m+1} = d_{m\delta+m+1} = 2\delta = c$. This graph example shows that the lower bound $d_{\sigma_m+m}+1$ in Theorem 1 cannot be replaced by $d_{\sigma_m+m+1}+1$. Thus, Theorem 1 is best possible in all respects.

**Corollary 1:** *Let $G$ be a graph with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. Then $c \geq d_{\delta+1} + 1$.*

The graph $K_\delta + (\delta + 1)K_1$ is $\delta$-connected with $d_1 = d_2 = ... = d_{\delta+1} = \delta$ and $d_{\delta+2} = 2\delta$. Since $c = 2\delta$ and $d_{\delta+2} + 1 = 2\delta + 1$, the bound $c \geq d_{\delta+1} + 1$ in Corollary 1 cannot be replaced by $c \geq d_{\delta+2} + 1$. Thus, Corollary 1 is best possible even for high connected graphs.

The next three statements can be obtained from Theorem 1 easily.

**Theorem 2:** *Let $G$ be a graph with $m$ connected components and degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. If $d_{\sigma_m+m} \geq p - 1$, then $c = p$.*

**Corollary 2:** *Let $G$ be a graph with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. If $d_{\delta+1} \geq p-1$, then $c = p$.*

**Theorem 3:** *Let $G$ be a graph with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. If $d_{\delta+1} \geq p-1$, then $G$ is hamiltonian.*

**Theorem 4:** *Let $G$ be a graph with degree sequence $\delta = d_1 \leq d_2 \leq ... \leq d_n$. If $d_{\delta+1} \geq n-1$, then $G$ is hamiltonian.*

## 2. Proofs

**Proof of Theorem 1**. Let $H_1, H_2, ..., H_m$ be the connected components of $G$ and let $\overrightarrow{P} = u\overrightarrow{P}v$ be a longest path in $H_1$. Clearly, $N(u) \subseteq V(P)$. Assume that

(a) $P$ is chosen in $H_1$ so that $d(u)$ is maximum.

Let $x_1, x_2, ..., x_t$ be the elements of $N(u)$ occurring on $\overrightarrow{P}$ in a consecutive order, where $t = d(u) \geq \delta$. Observe that for each $i \in \{1, 2, ..., t\}$,

$$x_i^- \overleftarrow{P} u x_i \overrightarrow{P} v$$

is a longest path in $H_1$, implying that

$$N(x_i^-) \subseteq V(P) \quad (i = 1, 2, ..., t).$$

By (a),

$$d(u) \geq d(x_i^-) \quad (i = 1, 2, .., t), \quad d(u) \geq d(v). \tag{1}$$

Put $C = u\overrightarrow{P}x_t u$. Clearly,

$$c \geq |V(C)| \geq t + 1 = d(u) + 1.$$

By (1),

$$d(u) \geq max\{d(x_1^-), d(x_2^-), ..., d(x_t^-), d(v)\},$$

implying that

$$c \geq d(u) + 1 \geq max\{d(x_1^-), d(x_2^-), ..., d(x_{d(u)}^-), d(v)\} + 1.$$

Analogously, for each $i \in \{1, 2, ..., m\}$, we can find pairwise distinct vertices $y_1^i, y_2^i, ..., y_{d(u_i)+1}^i$ in $H_i$ such that

$$c \geq \max\{d(y_1^i), d(y_2^i), ..., d(y_{d(u_i)+1}^i)\} + 1,$$

where $u_i \in V(H_i)$. Since the vertices

$$y_1^1, y_2^1, ..., y_{d(u_1)+1}^1,$$

$$y_1^2, y_2^2, ..., y_{d(u_2)+1}^2,$$

..........................

$$y_1^m, y_2^m, ..., y_{d(u_m)+1}^m$$

are all pairwise distinct, for some pairwise distinct vertices $z_1, z_2, ..., z_{d(u_1)+d(u_2)+...+d(u_m)+m}$ we have

$$c \geq \max\{d(z_1), d(z_2), ..., d(z_{d(u_1)+d(u_2)+...+d(u_m)+m})\} + 1$$

$$\geq \max\{d_1, d_2, ..., d_{d(u_1)+d(u_2)+...+d(u_m)+m}\} + 1$$

$$= d_{d(u_1)+d(u_2)+...+d(u_m)+m} + 1.$$

Observing also that $\{u_1, u_2, ..., u_m\}$ is an independent set of vertices, we obtain the desired bound $c \geq d_{\sigma_m+m} + 1$. ■

**Proof of Theorem 2**. By Theorem 1, $c \geq d_{\sigma_m+m} + 1 \geq p$. If $c \geq p + 1$, then clearly the cycle of order at least $p + 1$ contains a path with at least $p + 1$ vertices, contradicting the fact that the longest path in $G$ has exactly $p$ vertices. Hence, $c = p$. ■

**Proof of Theorem 3**. Let $G$ be a connected graph with $d_{\delta+1} \geq p - 1$. Since $m = 1$ and $d_{\sigma_m+m} \geq p - 1$, by Theorem 2, $c = p$. Let $\overrightarrow{C}$ be a longest cycle in $G$ of length $p$. If $p = n$, then $G$ is hamiltonian. Let $p < n$. Since $G$ is connected, we have $vu \in E(G)$ for some $v \in V(C)$ and $u \in G - C$. Then

$$uv\overrightarrow{C}v^-$$

is a path on $p + 1$ vertices, a contradiction. ■

Theorem 4 follows from Theorem 2 immediately.

## References


[1] J. A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[2] G. A. Dirac, "Some theorems on abstract graphs", *Proc. London, Math. Soc.*, vol. 2, pp. 69-81, 1952.


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## Գրաֆներում երկար ցիկլերի մասին աստիճանային հաջորդականությունների լեզվով

Մ. Քուլաքզյան

### Ամփոփում

Դիցուք $G$-ն $\delta$ նվազագույն աստիճան և $\delta = d_1 \leq d_2 \leq \ldots \leq d_n$ աստիճանային հաջորդականություն ունեցող $n$ գագաթանի գրաֆ է: Գրաֆի կապակցվածության բաղադրիչների քանակը կնշանակենք $m$-ով, ամենաերկար ցիկլի երկարությունը՝ $c$-ով, իսկ $s$ անկախ գագաթների աստիճանների նվազագույն գումարը՝ $\sigma_s$ -ով: Ներկա աշխատանքում ապացուցվում է, որ կամայական գրաֆում $c \geq d_{\sigma_m+m} + 1$: Ստացված գնահատականը ենթակա չէ բարելավման և ընդհանրացնում է 1952-ին Դիրակի կողմից ստացված $c \geq \delta + 1 = d_1 + 1$ գնահատականը:

## О длиннейших циклах графа в терминах последовательности степеней вершин

М. Кулакзян

### Аннотация

Пусть $\delta$=$d_1$≤$d_2$≤... ≤$d_n$ - последовательность степеней вершин $n$-вершинного графа $G$ с минимальной степенью $\delta$. Число компонент связности графа $G$ обозначается через $m$, длина длиннейшего цикла - через $c$, а минимальное число сумм степеней $s$ независимых вершин -через $\sigma_s$. В настоящей работе доказывается, что в любом графе $G$, $c \geq d_{\sigma_m+m} + 1$. Полученная оценка неулучшаема и обобщает оценку $c \geq \delta + 1 = d_1 + 1$ Дирака (1952).